1975

Kinematics and dynamics of elastic rods

William C. Davidon
Haverford College

AAPT PHYSICS EDUCATION
AMERICAN JOURNAL of PHYSICS

Kinematics and dynamics of elastic rods

William C. Davidon

Citation: Am. J. Phys. **43**, 705 (1975); doi: 10.1119/1.9718
View online: http://dx.doi.org/10.1119/1.9718
View Table of Contents: http://ajp.aapt.org/resource/1/AJPIAS/v43/i8
Published by the American Association of Physics Teachers

Kinematics and dynamics of elastic rods

William C. Davidon
Department of Physics
Haverford College
Haverford, Pennsylvania 19041
(Received 2 December 1974)

A simple macroscopic theory of elastic rods is presented in which all assumptions but one are consistent with both Newtonian mechanics and special relativity. The one distinguishing assumption is the inertial equivalence of energy. While invariance of the theory under Lorentz transformations is proved, all physical consequences—including the stress and velocity dependence of the length and inertial mass of a rod as well as the velocities of sound through it—are derived and can be tested in any one inertial frame. Exact wave solutions of the basic equations are obtained for an idealized elastic material in which the velocity of sound is independent of amplitude. These solutions are used to account for the kinematics and dynamics of accelerated rods, including the time-dependent processes which result in their overall Lorentz contraction.

1. INTRODUCTION

In Newtonian mechanics, the length of an elastic rod changes with stress but is independent of its velocity, while its linear momentum changes with velocity but is independent of stress. When the inertial equivalence of energy is considered, length and momentum each depend on both stress and velocity. While the velocity dependence of these and other quantities can be determined by symmetry under Lorentz transformations, symmetry alone does not determine the time-dependent stresses and velocities of different parts of a body during acceleration. For these, a more specific theory of elastic bodies is needed.

Theories based on Newtonian mechanics have long been used to describe the kinematics and dynamics of elastic bodies at nonrelativistic speeds.[1] Corresponding theories invariant under the Lorentz transformations of special relativity rather than the Galilean transformations of Newtonian physics, such as those presented by Synge,[2] Møller,[3] and others, have been used less frequently, probably because of their considerable complexity and the lack of experimental tests. Nevertheless, a study of the processes which take place in elastic bodies during acceleration, as described in any one inertial frame, can contribute to our understanding not only of elastic bodies but also of more general conservation laws and symmetry principles.

Here, we introduce a simple theory of moving elastic bodies and use it to account for the kinematics and dynamics of rods accelerated to any speed less than that of light. While symmetry of this theory under Lorentz transformations is proved, this need not be used in deriving the physical consequences of the theory in any one inertial frame. Of the five basic equations of this theory, four are consistent with both Newtonian mechanics and special relativity. The fifth, from which symmetry under Lorentz rather than Galilean transformations follows, is the inertial equivalence of energy; that is, to each quantity E of energy, there corresponds an inertial mass E/c^2. All aspects of the theory can be directly compared with their Newtonian counterparts by substituting zero for $1/c^2$.

To focus on the most essential physical ideas, we consider only one-dimensional rods, and we assume that all deformations occur elastically and adiabatically, with no thermal or hysteresis effects. To facilitate the physical interpretation of the theory and comparisons with its Newtonian limit, we leave the factor $1/c^2$ explicit and separately specify the space and time components of space–time vectors and tensors.

Throughout this paper, we choose a single though arbitrary inertial frame for the description of all physical processes, just as we choose a single though arbitrary set of units. We will consistently leave this choice implicit, and refer to the velocity of a particle or the momentum and kinetic energy of a body without repeating each time that these are with respect to the arbitrarily chosen inertial frame.

To ensure that no hidden assumptions are used in deriving our basic results, and to separate their precise statement and proofs from more informal and intuitive discussions of their physical significance, we give five essentially self-contained theorems and mark the end of their proofs with a square, □. We define ∂_x and ∂_t as differentiation with respect to space and time, and similarly ∂_s and ∂_v are differentiation with respect to stress S and velocity v. While we consider only one-dimensional rods, we choose lowercase greek letters for quantities which in the three-dimensional case are scalars, lowercase latin letters for quantities which generalize to 3-vectors, and uppercase latin letters for those which generalize to second-rank tensors or 3×3 matrices.

2. BASIC QUANTITIES AND ASSUMPTIONS

In the theory to be presented, there are just two independent quantities defined at every point in an elastic body at each time. These are the stress S and the velocity v. Figure 1 shows a rod moving with constant velocity v while subject to a uniform compressional stress S by a force $+S$ at its left end and $-S$ at its right end. In one dimension, the SI unit of stress is a newton, N, and in three dimensions it is a N/m². This theory will determine the time development of the stresses and velocities of all parts of an elastic body in terms of their values at any one time, the elastic properties of the material, and the values of any external forces.

In addition to the two independent quantities S and v, four dependent ones are basic:

(i) U, the strain, whose increment $U_1 - U_0$ at a point embedded in the material gives the factor $L_1/$

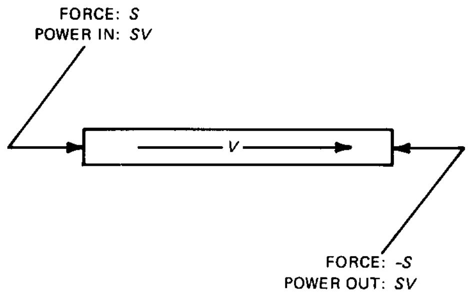

Fig. 1. A stressed rod in uniform motion. Balanced forces S and $-S$ transfer momentum at the rate S and energy at the rate Sv from the left to the right end of the rod.

$L_0 = \exp(U_1 - U_0)$ by which the distance L to each nearby point changes. It is dimensionless. There are various definitions for finite strain which all agree to first order. This one is chosen here since its time dependence is related to v by the simple differential equation (2.1).

(ii) ρ, the density of inertial mass, or simply the inertia density, including the inertial equivalent of all forms of energy, elastic and kinetic. The value of inertia density ρ when $v = 0$, as a function of stress alone, is defined as the mass density μ, that is, $\mu = \rho|_{v=0}$. The SI units of both ρ and μ are kg/m in one dimension and kg/m^3 in three.

(iii) g, the flux of inertial mass past a stationary point, as well the density of linear momentum, or simply momentum density. Its SI unit is kg/sec = N sec/m in one dimension, and kg/m^2 sec = N sec/m^3 in three.

(iv) T, the flux of linear momentum past a stationary point, with $T = S$ when $v = 0$. Its SI unit is a newton in one dimension and N/m^2 in three.

Our first assumption about these quantities is the only purely kinematic one,

$$(\partial_t + v\partial_x)U = \partial_x v. \tag{2.1}$$

This states that the total time rate of change of strain U at a point moving with the material equals the space rate of change of velocity v. Instead of assuming this, it could be derived from $(\partial_t + v\partial_x)U = \partial_t L/L$ and $\partial_t L = L\partial_x v$, where L is the distance between closely spaced points embedded in the material.

Our next two assumptions state that the time rate of change of inertial mass and linear momentum within any region equals their net flux into the region. When the only external forces are applied at the boundaries of the elastic body so that there are no sources or sinks for inertial mass or momentum in its interior, these assumptions give the homogeneous equations of continuity

$$\partial_t \rho + \partial_x g = 0 \tag{2.2}$$

and

$$\partial_t g + \partial_x T = 0. \tag{2.3}$$

Our last two assumptions express the flow of inertial mass and momentum as a sum of a convective part, equal to the density of the quantity times the velocity of the material, and a conductive part, equal to the flow past a point moving with the material,

$$T = gv + S \tag{2.4}$$

and

$$g = \rho v + Sv/c^2, \tag{2.5}$$

where $1/c^2 = 1.11265 \times 10^{-17}$ kg/J is the inertial equivalent of energy. The terms in Eqs. (2.4) and (2.5) have simple physical interpretations in the situation shown in Fig. 1. A rod under uniform stress S conducts linear momentum from one end to the other at the rate S, giving the second term on the right of Eq. (2.4). A moving rod under stress also transmits energy from one end to the other at the rate Sv, and the inertial equivalent Sv/c^2 of this energy flux gives the second term on the right of Eq. (2.5). This is the only place where the fundamental constant $1/c^2$ enters our basic assumptions, Eqs. (2.1)–(2.5).

3. STRESS AND VELOCITY DEPENDENCE OF QUANTITIES

Equations (2.4) and (2.5) alone give the momentum density g and the momentum flux T as functions of stress S, velocity v, and inertia density ρ. We now show that the stress and velocity dependence of ρ as well as of g, T, and the strain U are uniquely determined by our general assumptions, Eqs. (2.1)–(2.5), together with a constitutive equation giving the mass density $\mu = \rho|_{v=0}$ as a function of stress for the particular elastic material.

Theorem 1. For any differentiable function μ of S with $\mu + S/c^2 \neq 0$, there is just one set of functions ρ, g, T, and U of S and v which satisfy the boundary conditions $\mu = \rho|_{v=0}$ and $0 = U|_{s,v=0}$ as well as Eqs. (2.1)–(2.5) for all $\partial_x S$ and $\partial_x v$. These are

$$\rho = \frac{\mu + Sv^2/c^4}{1 - v^2/c^2}, \tag{3.1a}$$

$$g = \frac{\mu v + Sv/c^2}{1 - v^2/c^2}, \tag{3.1b}$$

$$T = \frac{\mu v^2 + S}{1 - v^2/c^2} \tag{3.1c}$$

and

$$U = \tfrac{1}{2}\ln(1 - v^2/c^2) - \int_0^\mu \frac{d\mu}{\mu + S/c^2}. \tag{3.2}$$

Proof. Use the chain rule to express the x and t derivatives of ρ, g, T, and U in terms of their S and v derivatives, for example, $\partial_t U = \partial_s U \partial_t S + \partial_v U \partial_t v$. Substitute these into Eqs. (2.1)–(2.3) to get three linear equations in $\partial_t S$, $\partial_t v$, $\partial_x v$, and $\partial_x v$,

$$\begin{bmatrix} \partial_s U & \partial_v U & v\partial_s U & v\partial_v U - 1 \\ \partial_s \rho & \partial_v \rho & \partial_s g & \partial_v g \\ \partial_s g & \partial_v g & \partial_s T & \partial_v T \end{bmatrix} \begin{bmatrix} \partial_t S \\ \partial_t v \\ \partial_x S \\ \partial_x v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

For this to have solutions for all $\partial_x S$ and $\partial_x v$, this 3×4 coefficient matrix must have rank 2 or less; that is, every 3×3 submatrix must have zero determinant. This gives two independent linear equations for $\partial_s U$ and $\partial_v U$. Use Eqs. (2.4) and (2.5) to express the S and v derivatives of g and T in terms of $\partial_s \rho$ and $\partial_v \rho$ and solve for $\partial_s U$ and $\partial_v U$ to get

$$\partial_s U = -\frac{\partial_s \rho}{\rho + S/c^2} + \frac{v^2/c^4}{(\rho + S/c^2)(1 - v^2/c^2)} \tag{3.3a}$$

and

$$\partial_v U = -\frac{\partial_v \rho}{\rho + S/c^2} + \frac{v/c^2}{1 - v^2/c^2}. \tag{3.3b}$$

These equations are integrable for U as a function of S and v if and only if the v derivative of the first equals the S derivative of the second, from which it follows that $\partial_v \rho/(\rho + S/c^2) = 2v/(c^2 - v^2)$. Integrate this with respect to v with the boundary condition $\mu = \rho|_{v=0}$ to get Eq. (3.1a). Substitute this ρ into Eqs. (2.4) and (2.5) to get Eqs. (3.1b) and (3.1c), and into Eqs. (3.3) to get

$$\partial_s U = -\frac{\partial_s \mu}{\mu + S/c^2} \tag{3.4a}$$

and

$$\partial_v U = -\frac{v/c^2}{1 - v^2/c^2}. \tag{3.4b}$$

Integrate these with the boundary condition $0 = U|_{s,v=0}$ to get Eq. (3.2). □

Combining Eq. (3.4a) with $\partial_s U = \partial_s L/L$ gives a first-order differential equation relating the stress dependence of length L and mass density μ,

$$\frac{\partial_s L}{L} = -\frac{\partial_s \mu}{\mu + S/c^2}. \tag{3.5}$$

This can be solved for the stress dependence of either L or μ when the other is known. Equation (3.5) can also be derived by considering the rate at which external forces do work on a rod. The rate at which the elastic energy E changes with stress is $\partial_s E = -S\partial_s L$. The total inertial mass μL of the rod at rest increases by the inertial equivalent of this elastic energy, so $\partial_s(\mu L) = \partial_s E/c^2 = -S\partial_s L/c^2$. Differentiating the product μL and solving for $\partial_s L$ gives Eq. (3.5).

Since $\partial_s^2 L = \partial_s(\partial_s L) = \partial_s(L\partial_s U) = \partial_s L \partial_s U + L\partial_s^2 U = L[(\partial_s U)^2 + \partial_s^2 U]$, it follows from Eq. (3.4a) that a rod satisfies Hooke's law $\partial_s^2 L = 0$ if and only if

$$(\mu + S/c^2)\partial_s^2 \mu = 2\partial_s \mu(\partial_s \mu + 1/2c^2). \tag{3.6}$$

The general solution for this second-order differential equation is

$$\mu = \frac{\mu_0 + S^2/2\mu_0 a_0^2 c^2}{1 - S/\mu_0 a_0^2},$$

where $\mu_0 = \mu|_{s=0}$ is the mass density of an unstressed rod at rest and $a_0 = (\partial_s \mu)^{-1/2}|_{s=0}$ will be shown to be the speed of sound through an unstressed rod at rest. This result can also be obtained by adding the inertial equivalent E/c^2 of the elastic energy $E = L_0 S^2/2\mu_0 a_0^2$ to the unstressed mass $\mu_0 L_0$ and dividing by the length $L = L_0(1 - S/\mu_0 a_0^2)$ of the rod under stress.

Theorem 1 shows that Eqs. (2.1)–(2.5) determine the velocity as well as the stress dependence of a rod. They give the $(1 - v^2/c^2)^{1/2}$ Lorentz contraction factor of an accelerating rod in any one inertial frame, rather than the ratio of lengths ascribed to the same rod by observers in different inertial frames. The stress and velocity dependence of the total inertial mass ρL and total momentum gL is also uniquely determined by Eqs. (2.1)–(2.5) with appropriate boundary conditions, since ρ and g as well as L are so determined.

4. SOUND VELOCITIES

In Newtonian physics, the inertial mass ρL of a rod is independent of both stress and velocity, its momentum gL changes with velocity but not stress, and its length L and strain U change with stress but not velocity; however, since the speed of sound a through matter at rest usually depends on stress, the velocities $w_\pm = v \pm a$ of sound through moving matter depend on both stress S and velocity v. When the inertial equivalence of energy is considered, all these quantities depend on both stress and velocity. Theorem 1 shows that Eqs. (2.1)–(2.5) together with the elastic properties of the material determine the stress and velocity dependence of ρ, g, T, and U, and now we show they determine the stress and velocity dependence of the velocities $w_\pm$ of sound as well.

We define a 2×2 matrix W which gives the time derivatives of S and v as linear combinations of their space derivatives,

$$\begin{bmatrix} \partial_t S \\ \partial_t v \end{bmatrix} = -\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} \partial_x S \\ \partial_x v \end{bmatrix}. \tag{4.1}$$

The eigenvalues $w_\pm$ of the matrix W are the velocities of sound, since when S and v depend only on $x - wt$, we have $\partial_t S = -w\partial_x S$ and $\partial_t v = -w\partial_x v$. The magnitude of $\partial_x S/\partial_x v = \partial_t S/\partial_t v$ for the eigenvectors of W is defined as the acoustic impedance of the material.[4]

Theorem 2. For any differentiable function μ of S with $\mu + S/c^2 \neq 0$ and $\partial_s \mu > 1/c^2$, Eqs. (2.1)–(2.5) with the boundary condition $\mu = \rho|_{v=0}$

uniquely determine the stress and velocity dependence of the matrix W in Eq. (4.1) for all $v^2 < c^2$, and it is

$$W = \frac{1}{\partial_s \mu - v^2/c^4} \times \begin{bmatrix} v(\partial_s \mu - 1/c^2) & \mu + S/c^2 \\ \dfrac{(1 - v^2/c^2)^2 \partial_s \mu}{\mu + S/c^2} & v(\partial_s \mu - 1/c^2) \end{bmatrix}. \quad (4.2)$$

The eigenvalues of this W are

$$w_\pm = \frac{v \pm a}{1 \pm av/c^2}, \quad (4.3)$$

where

$$a = (\partial_s \mu)^{-1/2}. \quad (4.4)$$

The ratio of the components of the eigenvectors of W is

$$\frac{\partial_t S}{\partial_t v} = \frac{\partial_x S}{\partial_x v} = \pm a \frac{\mu + S/c^2}{1 - v^2/c^2}. \quad (4.5)$$

Proof. Use the chain rule to express the x and t derivatives of ρ, g, and T in Eqs. (2.2) and (2.3) in terms of their S and v derivatives, and express the result as

$$\begin{bmatrix} \partial_s \rho & \partial_v \rho \\ \partial_s g & \partial_v g \end{bmatrix} \begin{bmatrix} \partial_t S \\ \partial_t v \end{bmatrix} = - \begin{bmatrix} \partial_s g & \partial_v g \\ \partial_s T & \partial_v T \end{bmatrix} \begin{bmatrix} \partial_x S \\ \partial_x v \end{bmatrix}.$$

This has a unique solution for $\partial_t S$ and $\partial_t v$ if and only if the determinant of their 2×2 coefficient matrix is not zero. Use Eqs. (3.1) to express the S and v derivatives of ρ, g, and T in terms of S, v, μ, and $\partial_s \mu$, so the determinant is $\partial_s \rho \partial_v g - \partial_s g \partial_v \rho = (\mu + S/c^2)(\partial_s \mu - v^2/c^4)(1 - v^2/c^2)^2$. Since the theorem assumes $\mu + S/c^2 \neq 0$, $\partial_s \mu > 1/c^2$, and $v^2 < c^2$, this determinant is nonzero and we can multiply on the left by the inverse matrix to obtain Eq. (4.1) with W given by Eq. (4.2).

Since $\partial_s \mu$ is assumed to be positive, Eq. (4.4) defines a positive number a. Obtain the eigenvalues $w_\pm$ of W as the roots of the characteristic equation for W, which in terms of a is

$$w^2(1 - a^2 v^2/c^4) - 2wv(1 - a^2/c^2) + v^2 - a^2 = 0.$$

This factors as

$$[w(1 + av/c^2) - v - a][w(1 - av/c^2) - v + a] = 0,$$

from which Eq. (4.3) follows. The ratio $\partial_t S/\partial_t v = \partial_x S/\partial_x v$ is determined by the eigenvalue equation

$$\begin{bmatrix} \partial_t S \\ \partial_t v \end{bmatrix} = -W \begin{bmatrix} \partial_x S \\ \partial_x v \end{bmatrix} = -w_\pm \begin{bmatrix} \partial_x S \\ \partial_x v \end{bmatrix}. \quad \square$$

Equation (4.3) for the sound velocities is usually obtained by making a Lorentz transformation by v on the velocities $\pm a$ of sound through stressed material at rest. Here, Eqs. (4.2)–(4.4) are obtained as consequences of Eqs. (2.1)–(2.5) alone. The matrix W defined by Eq. (4.1) can also be used to obtain Eqs. (3.4) for the stress and velocity dependence of strain U, since from Eqs. (2.1) and (4.1) alone it follows that $\partial_s U = [(vI - W)^{-1}]_{21}$ and $\partial_v U = [(vI - W)^{-1}]_{22}$, where I is the 2×2 identity matrix.

Since both the length L of a rod and the speed a of sound through it depend on stress, the period $T = 2L/a$ for a sound wave to make one round trip also depends on stress. Differentiating Eq. (4.4) with respect to stress gives

$$\frac{\partial_s a}{a} = -\frac{1}{2} \frac{\partial_s^2 \mu}{\partial_s \mu},$$

which with Eq. (3.5) gives

$$\frac{\partial_s T}{T} = \frac{\partial_s L}{L} - \frac{\partial_s a}{a} = -\frac{\partial_s \mu}{\mu + S/c^2} + \frac{1}{2} \frac{\partial_s^2 \mu}{\partial_s \mu}.$$

Thus the period of a sound wave going back and forth through a rod is independent of stress if and only if

$$(\mu + S/c^2) \partial_s^2 \mu = 2(\partial_s \mu)^2. \quad (4.6)$$

This agrees with Eq. (3.6) based on Hooke's law only in the Newtonian limit when the inertial equivalent of energy $1/c^2$ is neglected.

5. A GENERALIZATION OF HOOKE'S LAW

The stress dependence of mass density μ can be determined in several ways, for example, by measuring the length L of a rod or the speed a of sound through it as a function of stress and using Eq. (3.5) or (4.4). It can also be derived theoretically, at least in principle, by a quantum statistical model for the microscopic structure of the material. For most materials, only μ and $\partial_s \mu$ for small S can be determined without exceeding elastic limits, but it is conceptually and computationally convenient to extrapolate from these to an idealized constitutive equation which gives μ as a function of S for large stress as well. Two different such extrapolations are determined by Eqs. (3.6) and (4.6), the first based on the assumption that the change in length is proportional to stress, and the second on the stress independence of the period of a sound wave oscillating back and forth in a rod. We now consider a third extrapolation which is more useful than either of these for analyzing large-amplitude sound waves. It will be derived from the assumption that the velocity of a sound wave is independent of its amplitude. These three extrapolations are all equivalent to Hooke's law only in the Newtonian limit.

Consider two waves moving in the same direction, say

$+x$. The velocity of each is given by Eq. (4.3), $w_+ = (a + v)/(1 + av/c^2)$, where v is the velocity of the material and a is the speed of sound, which depends on stress. Each wave produces fluctuations in both velocity and stress and these must have exactly opposite effects on w_+ if neither wave is to change the velocity of the other. We now determine the stress dependence of mass density μ that is necessary and sufficient for this cancellation to occur.

Theorem 3. Let μ be a differentiable function of S and define a and $w_\pm$ by Eqs. (4.3) and (4.4). Then $\partial_x w_+ = 0$ for all $\partial_x S$ and $\partial_x v$ satisfying Eq. (4.5) if and only if μ satisfies

$$(\mu + S/c^2)\partial_s{}^2\mu = 2\partial_s\mu(\partial_s\mu - 1/c^2). \qquad (5.1)$$

The general solution of this equation is

$$\mu = \frac{\mu_0 + S/c^2}{1 - S(a_0{}^{-2} - c^{-2})/\mu_0}, \qquad (5.2)$$

where $\mu_0 = \mu|_{s=0}$ and $a_0 = a|_{s=0}$. This function is differentiable and satisfies $\mu + S/c^2 \neq 0$ for all S in the interval

$$-\frac{\mu_0 a_0 c}{1 + a_0/c} < S < \frac{\mu_0 a_0{}^2}{1 - a_0{}^2/c^2}. \qquad (5.3)$$

For these S,

$$a = (\partial_s\mu)^{-1/2} = [1 - S(a_0{}^{-2} - c^{-2})/\mu_0]a_0 \qquad (5.4)$$

and $0 < a < c$.

Proof. Set to zero the x derivative of $w_\pm$ given by Eq. (4.3) to get $(1 - a^2/c^2)\partial_x v \pm (1 - v^2/c^2)\partial_x a = 0$. Differentiate Eq. (4.4) with respect to x to get $\partial_x a = -a^3(\partial_s{}^2\mu)\partial_x S/2$, so that $\partial_x w_\pm = 0$ if and only if $(1 - a^2/c^2)\partial_x v = \pm(1 - v^2/c^2)a^3\partial_s{}^2\mu\partial_x S/2$. This holds for all $\partial_x S$ and $\partial_x v$ satisfying Eq. (4.5) if and only if $(1 - a^2/c^2) = a^4(\mu + S/c^2)\partial_s{}^2\mu/2$. Substitute $\partial_s\mu = a^{-2}$ from Eq. (4.4) into this to get the differential equation (5.1). Differentiation verifies that the μ of Eq. (5.2) is the general solution of this equation and gives the a of Eq. (5.4). As S approaches the positive limit $\mu_0 a_0{}^2/(1 - a_0{}^2/c^2)$, μ increases without bound and a goes to zero, while as S approaches the negative limit $-\mu_0 a_0 c/(1 + a_0/c)$, $\mu + S/c^2$ goes to zero and a approaches c. □

From Eqs. (3.5) and (5.2) we obtain the stress dependence of the length of a rod for this idealized constitutive equation:

$$L = L_0\frac{1 - (a_0{}^{-2} - c^{-2})S/\mu_0}{\{1 + [2 - (a_0{}^{-2} - c^{-2})S/\mu_0]S/\mu_0 c^2\}^{1/2}}. \qquad (5.5)$$

Combining Eqs. (5.2) and (5.5) gives the stress dependence of the inertial mass,

$$\mu L = \mu_0 L_0\frac{1 + S/\mu_0 c^2}{\{1 + [2 - (a_0{}^{-2} - c^{-2})S/\mu_0]S/\mu_0 c^2\}^{1/2}}, \qquad (5.6)$$

and combining Eqs. (5.4) and (5.5) gives the stress dependence of the period of acoustic oscillation,

$$T = \frac{2L}{a} = \left(\frac{2L_0}{a_0}\right)\left(\frac{1}{\{1 + [2 - (a_0{}^{-2} - c^{-2})S/\mu_0]S/\mu_0 c^2\}^{1/2}}\right).$$

This period and the inertial mass given by Eq. (5.6) become independent of stress in the Newtonian limit when 0 replaces $1/c^2$.

Figure 2 sketches the stress dependence of length given by Eq. (5.5), inertial mass given by Eq. (5.6), and the velocities of sound given by Eq. (5.4), together with the velocity dependence of these quantities and their Newtonian limits. The shapes of these curves depend only on the ratio a_0/c, and the exceptionally large value of $a_0/c = ½$ was used in making these graphs to exhibit more clearly the differences between these stress dependencies and their Newtonian limits. As the stress approaches its positive limit $\mu_0 a_0{}^2/(1 - a_0{}^2/c^2)$, the rod is compressed to arbitrarily small length, its inertial mass increases toward the limiting value $\mu_0 L_0/(1 - a_0{}^2/c^2)^{1/2}$ which it has when moving with velocity a_0 while unstressed, the velocities of sound approach zero, and the period of acoustic oscillations approaches its minimum value $(2L_0/a_0)(1 - a_0{}^2/$

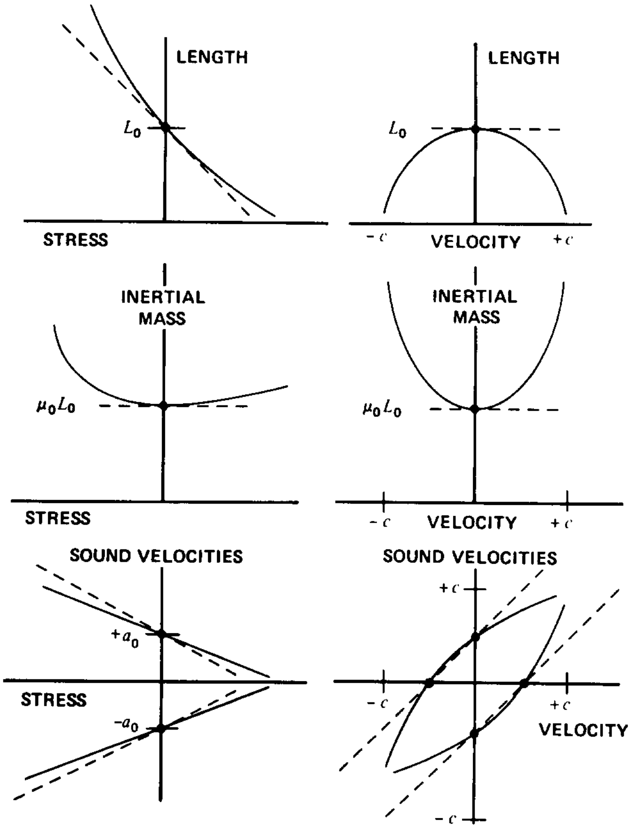

Fig. 2. Dependence of length L, inertial mass ρL, and the velocities of sound $w_\pm$ on the stress S and velocity v of an elastic rod satisfying a generalization of Hooke's law. Dashed lines show the Newtonian limits.

$c^2)^{1/2}$. As the stress approaches its negative limit $-\mu_0 a_0 c/(1 + a_0/c)$, the length and inertial mass of a rod increase without bound, the velocities of sound approach $\pm c$, and the period of acoustic oscillations increases without bound.

The idealized constitutive equation (5.2) can also be expressed as $\mu S = (\mu - S/c^2 - \mu_0)\mu_0 a_0^2/(1 - a_0^2/c^2)$. Since it follows directly from Eqs. (3.1) that $\rho T - g^2 = \mu S$ and $\rho - T/c^2 = \mu - S/c^2$, we obtain an idealized constitutive equation for just ρ, g, and T,

$$\rho T - g^2 = \frac{(\rho - T/c^2 - \mu_0)\mu_0 a_0^2}{1 - a_0^2/c^2}. \qquad (5.7)$$

Equations (2.2), (2.3), and (5.7) give three equations for just ρ, g, and T, and mathematically the simplicity of this idealized constitutive equation follows from the linearity of Eq. (5.7) in the velocity-independent quantities $\rho T - g^2$ and $\rho - T/c^2$.

Exact wave solutions of Eqs. (2.1)–(2.5) and (5.2) are now readily obtained, since for this idealized constitutive equation the velocity of a sound wave is independent of its amplitude and waveform.

Theorem 4. Let μ_0, a_0, and v_0 be numbers with $\mu_0 > 0$, $0 < a_0 < c$, and $v_0^2 < c^2$. Define $w = (a_0 + v_0)/(1 + a_0 v_0/c^2)$, let f be any function of $x - wt$ bounded by $-c/a_0 < f < 1$, and define $\phi = f(1 + a_0 v_0/c^2)^2/(1 - f)(1 - f a_0^2/c^2)$. Then a solution to Eqs. (2.1)–(2.5) and (5.2) is

$$S = \frac{f a_0^2 \mu_0}{1 - f a_0^2/c^2}, \qquad (5.8a)$$

$$v = \frac{v_0 + f a_0}{1 + f a_0 v_0/c^2}, \qquad (5.8b)$$

$$\mu = \mu_0/(1 - f), \qquad (5.8c)$$

$$U = \frac{1}{2}\ln\left(1 - \frac{v_0^2}{c^2}\right) + \ln\left(\frac{1 - f}{1 + f a_0 v_0/c^2}\right), \qquad (5.8d)$$

$$\rho = \frac{1 + \phi}{1 - v_0^2/c^2}\mu_0, \qquad (5.8e)$$

$$g = \frac{v_0 + \phi w}{1 - v_0^2/c^2}\mu_0, \qquad (5.8f)$$

$$T = \frac{v_0^2 + \phi w^2}{1 - v_0^2/c^2}\mu_0. \qquad (5.8g)$$

From each of these solutions, another can be obtained by substituting $-a_0$ for a_0 throughout.

Proof. Since ϕ is a function of f, it too depends only on $x - wt$ so $\partial_t \phi = -w\partial_x \phi$. Use this and Eqs. (5.8e)–(5.8g) to verify Eqs. (2.2) and (2.3). Verify Eqs. (2.4) and (2.5) by substituting for S, v, ρ, g, and T from Eqs. (5.8) and using the definition of ϕ in terms of f. Finally, use $\partial_t f = -w\partial_x f$ and Eqs. (5.8b) and (5.8d) to verify Eq. (2.1). □

The arbitrary function f in Theorem 4 can be interpreted as an invariant and dimensionless amplitude for the wave. Equations (5.8a) and (5.8b) can be solved for f in terms of S or v to give $f = S/(\mu_0 + S/c^2)a_0^2 = (v - v_0)/(1 - v v_0/c^2)a_0$. When f is small and the inertial equivalence of energy is neglected, then the fluctuations in strain produced by the wave equal $-f$, and $S = f\mu_0 a_0^2$, $v = v_0 + f a_0$, $\mu = \rho = \mu_0 + f\mu_0$, $g = \mu_0 v_0 + f\mu_0(a_0 + v_0)$, and $T = \mu_0 v_0^2 + f\mu_0(a_0 + v_0)^2$.

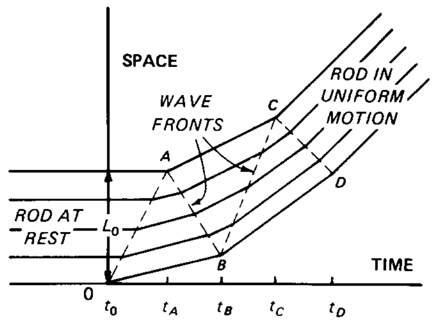

Fig. 3. Space–time diagram for an accelerating rod. A constant force is applied from time t_0 to time t_D at the $-x$ end of the rod, 0BD, while a sound wave goes back and forth.

6. ACCELERATING RODS

We now use Theorem 4 to describe quantitatively what happens when a constant force is applied to one end of an elastic rod which at $t_0 = 0$ is at rest and unstressed, as shown in the space–time diagram of Fig. 3. We assume an idealized material in which the velocity of sound is independent of amplitude.

Because the applied force is assumed constant, the total momentum of the rod increases linearly, but the distribution of this momentum is uneven and changes with time so that no part of the rod accelerates continuously as does its center of inertia. Instead, a sound wave starts at $t_0 = 0$ to travel down the rod, accelerating each part of the rod as it passes, leaving it with constant stress and velocity. At time t_A when the wave front is reflected from the other end of the rod, the entire rod is under a stress equal to the applied force and it is all moving with the same velocity. As the wave front returns, it leaves behind it an unstressed region moving at a higher velocity. In the adiabatic approximation, the wave continues to go back and forth, the stress at each point of the rod alternates between zero and the applied force, and the velocity at each point increases in a stepwise fashion. If the applied force is removed when the entire rod is unstressed, as at time t_B or t_D in Fig. 3, there will be no further changes in stress or velocity.

For a quantitative description of the first pass of the wave, we use Theorem 4 with $v_0 = 0$, $f = 0$ for $x - wt > 0$, and $f = v/a_0$ for $x - wt < 0$, where $w = a_0$ is the velocity of the first wave. Substituting into Eqs. (5.8) gives the quantities at time t_A, after this wave has passed:

$$S_A = \frac{a_0 v \mu_0}{1 - a_0 v/c^2},$$

$$v_A = v,$$

$$U_A = \ln(1 - v/a_0),$$

$$\rho_A = \frac{[1+(v-a_0)v/c^2]\mu_0}{(1-v/a_0)(1-a_0v/c^2)},$$

$$g_A = \frac{v\mu_0}{(1-v/a_0)(1-a_0v/c^2)},$$

$$T_A = \frac{a_0v\mu_0}{(1-v/a_0)(1-a_0v/c^2)}.$$

The length L_A of the rod at time t_A is $L_A = L_0 \exp(U_A - U_0 = L_0(1 - v/a_0)$. The time required for the first wave to traverse the rod is $t_A = L_0/a_0$.

For the return wave, we match the stress and velocity at time t_A, using $v_0 = 2v/(1 + v^2/c^2)$, $f = -v/a_0$ for $x - wt < L_0 - wt_A$, and $f = 0$ for $x - wt > L_0 - wt_A$, where $w = (v_0 - a_0)/(1 - a_0v_0/c^2)$ is the velocity of the return wave. Substituting into Eqs. (5.8) gives the quantities at time t_B, after the first acceleration cycle:

$$S_B = 0,$$

$$v_B = \frac{2v}{1+v^2/c^2}$$

$$U_B = \ln\left(\frac{1-v^2/c^2}{1+v^2/c^2}\right),$$

$$\rho_B = \mu_0 \frac{(1+v^2/c^2)^2}{(1-v^2/c^2)^2},$$

$$g_B = 2\mu_0 \frac{v(1+v^2/c^2)}{(1-v^2/c^2)^2},$$

$$T_B = 4\mu_0 \frac{v^2}{(1-v^2/c^2)^2}.$$

The length L_B of the rod at time t_B is $L_B = L_A \exp(U_B - U_A) = L_0 \exp(U_B - U_0) = L_0(1 - v^2/c^2)/(1 + v^2/c^2)$. The time $t_B - t_A$ is determined by $(t_B - t_A)(v - w_{AB}) = L_A = L_0(1 - v/a_0)$ and is $t_B - t_A = t_A(1 + v^2/c^2 - 2a_0v/c^2)/(1 - v^2/c^2)$. In the Newtonian limit, $t_B - t_A = t_A$ and each traversal by the wave takes the same time. When the inertial equivalence of energy is considered, then when the rod is being pushed within its elastic limits, $0 < v < a_0$ and $t_B - t_A < t_A$ while, when it is pulled, $-c < v < 0$ and $t_B - t_A > t_A$.

Alternate kinematic and dynamic arguments can be used to check the results obtained with Theorem 4. For example, since the external force $F = S$ increases the total momentum of the rod at the rate S and its total inertial mass at the rate Fv/c^2, conservation laws alone give $gL = St$ and $\rho L = \mu_0 L_o + Svt/c^2$ at times $t_0 = 0$, t_A, and t_B.

Subsequent acceleration cycles can either be analyzed directly as was the first, or they can be obtained from the first by active Lorentz transformations to be considered in the next section. As long as the applied force remains constant, the magnitude of f is the same in every cycle, $f = S/(\mu_0 + S/c^2)a_0^2$. The velocity v of the rod when all of it is stressed and the velocity v_f at the end of each acceleration cycle are given in terms of fa_0 and the velocity v_i at the beginning of each cycle by $v = (fa_0 + v_i)/(1 + fa_0v_i/c^2)$ and $v_f = (fa_0 + v)/(1 + fa_0v/c^2)$. The ratio of the unstressed lengths of the rod before and after each cycle is $L_f/L_i = (v_f - v)/(v - v_i) = (1 - fa_0v/c^2)/(1 + fa_0v/c^2) = (1 - v_f^2/c^2)^{1/2}/(1 - v_i^2/c^2)^{1/2}$. The time to complete each cycle is

$$T = \frac{1 - fa_0^2/c^2}{(1 - f^2a_0^2/c^2)^{1/2}(1 - v^2/c^2)^{1/2}} \frac{2L_0}{a_0}.$$

When $v = 0$, this reduces to the period of acoustic oscillations obtained in Sec. 5 for a rod at rest.

Here, as in our previous derivation of the Lorentz contraction, the familiar $(1 - v^2/c^2)^{1/2}$ factor in the denominator gives the increase in the period of oscillation for an accelerating rod as described in one inertial frame, rather than the ratio of the periods ascribed to the same oscillation by observers in different inertial frames.

7. ACTIVE LORENTZ TRANSFORMATIONS

From each solution to the basic equations (2.1)–(2.5), others can be obtained by symmetry. Here we consider only active symmetry transformations which change the state of physical systems, rather than passive transformations which change only how a given physical situation is described.

More precisely, to each state A of a physical system and each space–time point P, an active symmetry transformation l associates a new physical state lA and a new space–time point lP. For other quantities, such as the vetial frame, coordinate system, units, measurement apparatus, etc. Certain measurements on state A in the neighborhood of any space–time point P—for example, stress and velocity—uniquely determine the results of all measurements on the new state lA made in the neighborhood of the new space–time point lP.

Consider, for example, rod I in the space–time diagram of Fig. 4. It initially moves at velocity $-v$, and after an elastic collision with a fixed object, it rebounds with velocity $+v$. An active Lorentz transformation by $+v$ changes this situation to the one analyzed in some detail in the last section, in which rod II, initially at rest, is struck by an object moving with constant velocity $+v$.

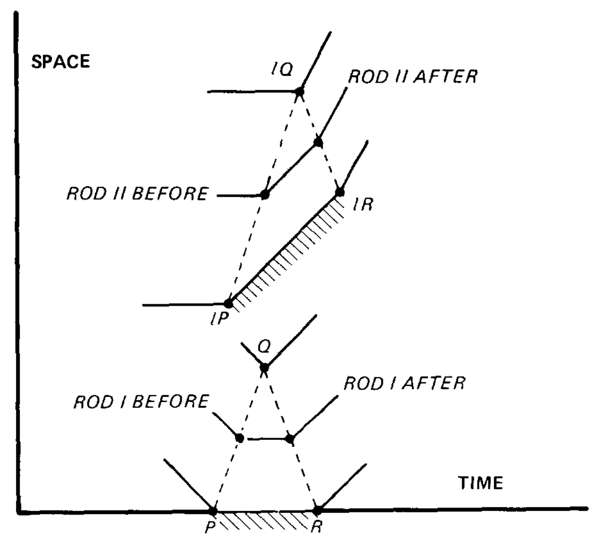

Fig. 4. Space–time diagram for two elastic collisions related by an active Lorentz transformation which changes the state of rod I to that of rod II and shifts space–time points such as P to lP.

For some quantities, such as stress, measurements made on state A at the space–time point P give the same result $S_A|_P$ as measurements made on new state lA at the space–time point lP. For other quantities, such as the velocity v of the material, $v_A|_P$ determines $v_{lA}|_{lP}$ but these are not equal.

We now consider how a Lorentz transformation l by a velocity shift u changes each of the basic quantities of this theory. The coordinates of the points P and lP, in the same coordinate system, are related by

$$x_{lP} = \frac{x_P + t_P u}{(1 - u^2/c^2)^{1/2}} \tag{7.1a}$$

and

$$t_{lP} = \frac{t_P + x_P u/c^2}{(1 - u^2/c^2)^{1/2}}. \tag{7.1b}$$

If $-u$ is substituted for u, these equations give the inverse transformation l^{-1}, and if 0 is substituted for the inertial equivalent of energy $1/c^2$, these give a Galilean transformation.

Theorem 5. Let S_A, v_A, ρ_A, T_A and U_A be one solution to Eqs. (2.1)–(2.5) and let u be any number with $u^2 < c^2$. Then a new solution S_{lA}, v_{lA}, ρ_{lA}, g_{lA}, T_{lA}, and U_{lA} of these same equations is given by

$$S_{lA}|_{lP} = S_A|_P, \tag{7.2a}$$

$$v_{lA}|_{lP} = \left.\frac{u + v_A}{1 + uv_A/c^2}\right|_P, \tag{7.2b}$$

$$\rho_{lA}|_{lP} = \left.\frac{\rho_A + 2g_A u + T_A u^2/c^4}{1 - u^2/c^2}\right|_P, \tag{7.2c}$$

$$g_{lA}|_{lP} = \left.\frac{\rho_A u + g_A(1 + u^2/c^2) + T_A u/c^2}{1 - u^2/c^2}\right|_P, \tag{7.2d}$$

$$T_{lA}|_{lP} = \left.\frac{\rho_A u^2 + 2g_A u + T_A}{1 - u^2/c^2}\right|_P, \tag{7.2e}$$

$$U_{lA}|_{lP} = U_A + \ln \left.\frac{(1 - u^2/c^2)^{1/2}}{1 + uv_A/c^2}\right|_P, \tag{7.2f}$$

where the coordinates of points P and lP are related by Eqs. (7.1).

Proof. Use Eqs. (7.2) alone to derive

$$(gv + S - T)_{lA}|_{lP} = \left.\frac{[gv + S - T + (\rho v + Sv/c^2 - g)u]_A}{1 + uv_A/c^2}\right|_P$$

and

$$(\rho v + Sv/c^2 - g)_{lA}|_{lP} = \left.\frac{[\rho v + Sv/c^2 - g + (gv + S - T)u/c^2]_A}{1 + uv_A/c^2}\right|_P.$$

From these it follows that, if S, v, ρ, g, and T for state A satisfy Eqs. (2.4) and (2.5), they do for state lA as well.

When considering the differential equations (2.1)–(2.3), it is convenient to extend the definition of the Lorentz transformation l so that it permutes functions over space–time as well as space–time points. For any function f of x and t, define a new function lf by $lf|_{lP} = f|_P$ for all P; that is, the value of the new function lf at the new point lP always equals the value of the old function f at the old point P. It follows immediately that l preserves sums and products of functions; that is, $l(f + g) = lf + lg$ and $l(fg) = (lf)(lg)$ for any two functions f and g of x and t. The transformation l does not commute with differentiation with respect to x and t, since Eqs. (7.1) and the chain rule for differentiation imply the operator identities

$$\partial_t l = \frac{l(\partial_t - u\partial_x)}{(1 - u^2/c^2)^{1/2}} \tag{7.3a}$$

and

$$\partial_x l = \frac{l(\partial_x - u\partial_t/c^2)}{(1 - u^2/c^2)^{1/2}}. \tag{7.3b}$$

Use these and Eqs. (7.2) alone to derive

$$\partial_t \rho_{lA} + \partial_x g_{lA} = \frac{l[\partial_t \rho_A + \partial_x g_A + (\partial_t g_A + \partial_x T_A)u/c^2]}{(1 - u^2/c^2)^{1/2}}$$

and

$$\partial_t g_{lA} + \partial_x T_{lA} = \frac{l[\partial_t g_A + \partial_x T_A + (\partial_t \rho_A + \partial_x g_A)u]}{(1 - u^2/c^2)^{1/2}}.$$

Hence, when ρ, g, and T for state A satisfy Eqs. (2.2) and (2.3), they do so for state lA as well. Finally, use Eqs. (7.2) and (7.3) alone to derive

$$(\partial_t + v_{lA}\partial_x)U_{lA} - \partial_x v_{lA} = l\left(\frac{(\partial_t + v_A\partial_x)U_A - \partial_x v_A}{1 + uv_A/c^2}\right)\left(1 - \frac{u^2}{c^2}\right)^{1/2},$$

so that, when v_A and U_A satisfy Eq. (2.1), v_{lA} and U_{lA} do so as well.

For a simple and useful example, we make a Lorentz transformation by v on a stressed rod at rest, so that $S_A = S$, $v_A = 0$, $\rho_A = \mu$, $g_A = 0$, $T_A = S$ and $U_A = -\int_0^\mu d\mu/(\mu + S/c^2)$, and the final state lA is described by Eqs. (3.1) and (3.2).

The transformation of most of the basic quantities in this theory are familiar ones; S and μ are scalars, ρ, g, and T are the components of a symmetric rank two space–time tensor, and v is the ratio between the components of a space–time vector. However, the strain U and the length L of a rod are not as simply related to space–

time vectors and tensors. While quantities such as ρ, g, and T which transform linearly can always be decomposed into scalars, vectors, and tensors of other rank, this is not the case for quantities such as strain U which undergo nonlinear transformations.

The proof of Theorem 5 shows that transforming any solution of Eq. (2.1) alone gives a new solution, whether or not the old one satisfies any of the other basic equations. Hence, the set of all solutions or the solution set for Eq. (2.1) is invariant under Lorentz transformations even though this equation is not written in covariant form. While none of the other four basic equations (2.2)–(2.5) has a solution set that is invariant under Lorentz transformations, the intersection of those for Eqs. (2.2) and (2.3) is invariant, as is the intersection of solution sets for Eqs. (2.4) and (2.5).

8. SUMMARY AND CONCLUSIONS

The macroscopic theory of elastic rods that has been presented assumes five basic equations, (2.1)–(2.5), relating stress S, velocity of the material v, strain U, inertial density ρ, inertial flux and momentum density g, and momentum flux T. Section 2 presents these equations and the intuitive physical significance of each. Theorem 1 shows that the stress and velocity dependence of ρ, g, T, and U is uniquely determined by these assumptions together with appropriate boundary conditions. From these, the usual relativistic results for the velocity dependence of the length and inertial mass of a rod are derived.

Theorem 2 shows that, in this theory, the values and space derivatives of S and v at any point uniquely determine their time derivatives, and from this the velocities $(v \pm a)/(1 \pm av/c^2)$ of sound through moving matter are derived, where a is the speed of sound through matter at rest. Theorem 3 determines the stress dependence of mass density that is necessary and sufficient for the velocity of sound waves to be independent of amplitude. In Newtonian mechanics, this reduces to Hooke's law. Theorem 4 gives exact, finite-amplitude wave solutions for these equations, with the values for S, v, ρ, g, T, and U at all parts of the wave. Theorem 5 shows that from any solution of the basic equations, regardless of the elastic properties of the material, new ones can be obtained by active Lorentz transformations which change the physical situation as described in any one inertial frame. This establishes the Lorentz invariance of the theory even though the basic equations are not written in manifestly covariant form.

This approach to the study of elastic bodies has certain advantages even in situations when relativistic effects are small. It provides a detailed picture for the localized densities and flows of conserved quantities in elastic objects subject to stress and acceleration, and this often contributes to an understanding of physical processes at any speed. It describes different physical processes in a single, arbitrarily chosen inertial frame by using just one coordinate system and set of units rather than transforming among equivalent descriptions of the same physical processes.

Many of our results are certainly well known from special relativity, including the velocity dependence of the length, inertia, and period of oscillation of an accelerating rod, and the relativistic velocity addition formula. All that is new for these here is that they are derived in any one inertial frame from assumptions in which the fundamental constant $1/c^2$ enters only in the one term of Eq. (2.5), where it associates a momentum density Sv/c^2 to the energy flux Sv.

In another paper,[5] certain other occurrences of the speed of light c in physics, including those in Maxwell's equations of electromagnetism, were also derived from assumptions in which the only $1/c^2$ terms were identified with the inertial equivalence of energy. These results support the conjecture that with adequate specific theories all occurrences of the speed of light c in physics can be derived from the inertial equivalent of energy $1/c^2$.

In addition to providing an alternate derivation of well-known results and a more detailed model for following relativistic processes in any one inertial frame, this theory gives certain new results. For example, it shows that certain conditions which each determine the stress dependence of mass density, while all equivalent to Hooke's law in Newtonian mechanics, give different generalizations of it when the inertial equivalence of energy is taken into account. Three of these are (i) deformation proportional to stress, $\partial_s{}^2L = 0$ or [Eq. (3.6)]

$$(\mu + S/c^2)\partial_s{}^2\mu = 2\partial_s\mu(\partial_s\mu + 1/2c^2),$$

(ii) stress independence of the period of acoustic oscillations, $\partial_s(L/a) = 0$ or [Eq. (4.7)],

$$(\mu + S/c^2)\partial_s{}^2\mu = 2(\partial_s\mu)^2,$$

and (iii) amplitude independence of the velocity of sound waves [Eq. (5.1)],

$$(\mu + S/c^2)\partial_s{}^2\mu = 2\partial_s\mu(\partial_s\mu - 1/c^2).$$

This theory also gives more information about the dynamics of elastic deformation and sound wave propagation than is determined by Lorentz invariance alone.

While only one-dimensional rods have been considered in this paper, the basic equations (2.1)–(2.5) generalize to three space dimensions. In this case, additional constitutive equations are needed to relate shear stresses to the deformations they produce in each elastic material, since the stress dependence of mass density no longer uniquely determines the elastic properties of material as it does for the one-dimensional case.

ACKNOWLEDGMENTS

It is a pleasure to thank Jerry Gollub for studying an earlier draft of this paper and for his helpful suggestions, particularly in clarifying several aspects of the physical interpretation of this theory.

[1]A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Dover, New York, 1944), 4th ed.

[2]J. L. Synge, *Relativity: the Special Theory* (North-Holland, Amsterdam, 1965), 2nd ed., pp. 173–181.

[3]C. Møller, *The Theory of Relativity* (Clarendon, Oxford, England, 1972), 2nd ed., pp. 260–301.

[4]P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (McGraw-Hill, New York, 1968), p. 259.

[5]W. C. Davidon, Found. Phys. (to be published).