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A Multiple Exchange Property for Bases

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A MULTIPLE EXCHANGE PROPERTY FOR BASES

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ABSTRACT. Let X and Y be bases of a combinatorial geometry G, and let A be any subset of X. Then there exists a subset B of Y with the property that $(X-A)\cup B$ and $(Y-B)\cup A$ are both bases of G.

I. **Introduction.** This paper is an outgrowth of a number of recent efforts to extend the methods of both linear algebra and classical invariant theory to the study of combinatorial geometries ([**3**], [**4**], [**5**], [**6**], [**7**]). One result of such efforts will, hopefully, be a completely satisfactory coordinatization theory for geometries—one which will include general techniques for automatically translating linear arguments into combinatorial ones. In spite of much encouraging work in this direction—and many interesting results—the full story apparently remains to be told.

As a result, the gap between "linear" and "nonlinear" combinatorial geometries sometimes seems embarrassingly large. There exist results which are easy to derive for linear geometries, using determinants or other techniques of linear algebra—but which are apparently much more difficult to prove by direct combinatorial methods.

This paper is devoted to the following example:

THEOREM. *Let X and Y be bases of a geometry G. Then for any subset $A\subseteq X$, there exists a subset $B\subseteq Y$ with the property that $(X-A)\cup B$ and $(Y-B)\cup A$ are both bases of G.*

The case $|A|=1$ is a slight strengthening of the fact taken by Whitney [**7**] as the defining property for bases. It is easily proved by elementary arguments (see [**2**]).

If G is linearly representable— that is, if the points of G can be represented as points in a vector space V over a field F in such a way that dependence in G corresponds to linear dependence in V—then the result

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is an immediate consequence of the Laplace expansion theorem for determinants. The argument is as follows:

Suppose that G has dimension n. We choose the elements of the basis Y as coordinate vectors, and assume that the points of G are represented accordingly as n-tuples over F. For any set $S \subseteq G$ of size n, we define $M(S)$ to be the $n \times n$ matrix whose columns are the vectors in S. Since X is a basis, the matrix $M(X)$ is nonsingular. Applying the Laplace expansion theorem to the set A of columns of $M(X)$, we obtain

$$\det M(X) = \sum_{B \subseteq Y} \pm \det M((X - A) \cup B) \det M((Y - B) \cup A).$$

Since $\det M(X) \neq 0$, some term on the right must be nonzero, and the result follows.

We remark that our combinatorial proof of this fact (given below) is not totally without interest in the linear case since it can easily be translated into an algorithm for actually finding the set B. Purely combinatorial versions of the exchange theorem can be obtained from the classical examples of combinatorial geometries—for example, if "bases" are replaced by spanning trees of a graph or maximal transversals of a family of sets. In these cases our proof provides a constructive method for carrying out the exchange.

2. Proof of the theorem.

For the basic facts about combinatorial geometries, we refer the reader to [**1**] or [**2**].

We begin with a few elementary lemmas.

LEMMA 1. *Let X and Y be bases of G, and let $x \in X$. Let d be the copoint spanned by $X - x$ and let C be the unique circuit obtained by adding x to Y. Then for any $y \in Y$, $(X-x) \cup y$ and $(Y-y) \cup x$ are both bases $\Leftrightarrow y \nleq d$ and $y \in C$.*

PROOF. Immediate.

LEMMA 2. *Suppose $y_1, \cdots, y_{n-1}$ are independent and span a copoint d_0. Let $y_1', \cdots, y_k'$ be points such that for each i,*

(1) $y_1' \vee y_2' \vee \cdots y_i' \vee y_{i+1} \vee \cdots \vee y_{n-1}$ is a copoint, say d_i.

(2) $d_0 \neq d_1 \neq \cdots \neq d_k$.

Then $\bigwedge_{j=0}^{k} d_j = y_{k+1} \vee \cdots \vee y_{n-1}$.

PROOF. Immediate, by induction on k.

LEMMA 3. *Suppose $C_1, \cdots, C_m$ are circuits with the property that for each $i = 2, \cdots, n$ there exists an element $y_i \in C_i - \bigcup_{j=1}^{i-1} C_j$. Then*

$$r\left(\bigcup_1^m C_i\right) \leqq \left|\bigcup_1^m C_i\right| - m.$$

PROOF. List the elements of $\bigcup_1^m C_i$ in order, beginning with C_1, C_2, etc. Then there are at least m distinct elements which depend on their predecessors.

THEOREM. *Let X and Y be bases of a geometry G. Then for any subset $A \subseteq X$, there exists a subset $B \subseteq Y$ with the property that $(X-A)\cup B$ and $(Y-B)\cup A$ are both bases of G.*

PROOF. Let $X=\{x_1, \cdots, x_n\}$ and $Y=\{y_1, \cdots, y_n\}$. We proceed by induction on the size of A. Assume that the theorem holds if $|A|=k$, and suppose now that $A=\{x_1, \cdots, x_{k+1}\}$. By assumption, we can exchange $x_1, \cdots, x_k$ for some subset of Y, which we denote by $y_1, \cdots, y_k$. Thus

$$X' = y_1, \cdots, y_k, x_{k+1}, \cdots, x_n \quad \text{and}$$

$$Y' = x_1, \cdots, x_k, y_{k+1}, \cdots, y_n$$

are both bases. The idea of the proof is as follows: we attempt to exchange x_{k+1} for one of the y's in Y'. If this is impossible, we exchange certain y's in X' for y's in Y' until it becomes possible. The proof consists of showing that an appropriate sequence of switches can always be found.

Technically, it turns out that we cannot always switch y's in such a way that both sets remain bases. In our proof, we require only that the set $X'-x_{k+1}$ have rank $n-1$ at each step. We use the following notation:

$$X' = U_X \cup U_Y, \quad U_X = \{x_{k+1}, \cdots, x_n\}, \quad U_Y = \{y_1, \cdots, y_k\},$$

$$Y' = V_X \cup V_Y, \quad V_X = \{x_1, \cdots, x_k\}, \quad V_Y = \{y_{k+1}, \cdots, y_n\},$$

$$d_0 = \bigvee (X' - x_{k+1}) \quad \text{(the copoint obtained by removing } x_{k+1} \text{ from } X'),$$

$$\begin{aligned} C_0 &= c(x_{k+1}, Y') \quad \text{(the circuit obtained by adding } x_{k+1} \text{ to } Y') \\ &= x_{k+1} \cup C_X^0 \cup C_Y^0, \quad C_X^0 \subseteq X, \quad C_Y^0 \subseteq Y, \end{aligned}$$

$$\begin{aligned} C_i &= c(y_i, Y') \quad \text{(defined for } y_i \in U_Y) \\ &= y_i \cup C_X^i \cup C_Y^i, \quad C_X^i \subseteq X, \quad C_Y^i \subseteq Y. \end{aligned}$$

If there exists a $y \in C_y^0$ with $y \nleq d_0$, then we can stop immediately for, by Lemma 1, $X'-x_{k+1}\cup y$ and $Y'-y\cup x_{k+1}$ are both bases. So from now on we assume that $y \leqq d_0$ for all $y \in C_y^0$.

We define an *admissible sequence of exchanges* ("admissible sequence" for short) to be a sequence of pairs

$$(y_1, y_1'), \cdots, (y_p, y_p')$$

satisfying the following conditions:

(i) $y_i \in U_{Y'}$, $y_1' \in V_Y - C_0$, $i=1, \cdots, p$.

(ii) For $i=1, \cdots, p$, $\bigvee (X'-x_{k+1}-y_1 \cdots -y_i \cup y_i' \cup \cdots \cup y_i')$ is a copoint, hereafter denoted by $d_{12\cdots i}$.

(iii) For $i=1, \cdots, p$, $Y'-y_1' \cdots -y_i' \cup y_1 \cup \cdots \cup y_i$ is a basis, hereafter denoted by $Y_{12\cdots i}'$.

(iv) $d_0 \neq d_1 \neq d_{12} \neq \cdots \neq d_{12\cdots p}$.

Thus admissible sequences are sequences of exchanges of elements in U_Y for elements in $V_Y - C_0$ which preserve a copoint-basis pair and have the property that each new copoint so obtained is distinct from the previous one. Condition (iv) is equivalent to requiring that $y_1' \nleqq d_0$ and $y_{i+1}' \nleqq d_{1\cdots i}$ for $i=1, \cdots, p-1$.

We define

$Q = \{d \mid d = d_{1\cdots p}$ for some admissible sequence$\} \cup \{d_0\}$,

$S = \{y_i \in U_Y \mid$ there exists an admissible sequence ending in $(y_i, y_i')\}$,

and

$T = U_Y - S$.

Thus T is the set of elements in U_Y which are never switched in any admissible sequence. It may of course be empty.

The rest of the proof consists of showing that there exists some admissible sequence which leads to a situation in which x_{k+1} can be exchanged. More precisely, we show that for some sequence there exists $y \in C_Y^0$ with the property that

$$X' - x_{k+1} - y_1 - \cdots - y_p \cup y_1' \cdots y_p' \cup y$$

and

$$Y_{12\cdots p}' - y \cup x_{k+1}$$

are both bases. To verify this, assume the contrary—that is, no admissible sequence leads to the situation just described. We complete the proof with a series of seven observations, leading to a contradiction:

(1) $y \leqq d$ for all $y \in C_Y^0$ and all $d \in Q$.

PROOF. We have already shown that $y \leqq d_0$ for all $y \in C_Y^0$. Suppose now that $d=d_{12\cdots p}$. Then $C_0=c(x_{k+1}, Y')=c(x_{k+1}, Y_{12\cdots p}')$, since condition (i) guarantees that no y_i' removed from Y' is in C_0. This means that the elements exchangeable for x_{k+1} in Y' and $Y_{12\cdots p}'$ are identical. If there exists $y \in C_Y^0$ with $y \nleqq d_{12\cdots p}$, then $X'-x_{k+1}-y_1-\cdots-y_p \cup y_1' \cup \cdots \cup y_p' \cup y$ and $Y_{12\cdots p}'-y \cup x_{k+1}$ are both bases. Since this was assumed not to occur, the conclusion follows.

(2) Let $y_j \in T$. Then $y \leqq d$ for all $y \in C_Y^j$ and all $d \in Q$.

PROOF. It is clear that $y \leqq d_0$ for all $y \in C_Y^j$, since otherwise (y_j, y) is an admissible sequence of length one. (Note that $y \nleqq d_0$ implies $y \notin C_0$, so that (i) is satisfied.) If $d=d_{12\cdots p}$, we may suppose that $y \leqq d'$ for all $y \in C_Y^j$ and $d'=d_0, d_1, \cdots, d_{12\cdots(p-1)}$. As before, $C_j=c(y_j, Y')=c(y_j, Y'_{12\cdots p})$, since $y'_1 \nleqq d_0$ and $y'_{i+1} \nleqq d_{12\cdots i}$ for $i=1, \cdots, p-1$, and so every y'_i removed is outside of C_j (by the inductive hypothesis). If there exists $y \in C_Y^j$ with $y \nleqq d_{12\cdots p}$, then $(y_1, y'_1), \cdots, (y_p, y'_p), (y_j, y)$ is an admissible sequence, contradicting the assumption that $y_j \in T$. (Again, $y \notin C_0$ since $y \leqq d_{12\cdots p}$ for all $y \in C_Y^0$, by (1).) This completes the proof.

(3) Let $\beta = \bigwedge_{d \in Q} d$. Then $y \leqq \beta$ for every $y \in Y$ which appears among the circuits C_0 and C_i, $y_i \in T$.

PROOF. This is an immediate consequence of (1) and (2), and the definition of T.

We pause briefly at this point to sketch the idea behind the rest of the proof. We will show that (3) is impossible because too many x's "depend" on the y's in β. In fact, the remaining steps show that adding certain x's to β results in a flat of dimension $<n$ which spans all the x's, contradicting the fact that X is a basis.

(4) $\beta = \bigvee (X' - x_{k+1} - S)$.

PROOF. By Lemma 2, $d_0 \wedge d_1 \wedge \cdots \wedge d_{12\cdots p} = \bigvee (X' - x_{k+1} - y_1 - \cdots - y_p)$ for any admissible sequence $(y_1, y'_1), \cdots, (y_p, y'_p)$. The result follows immediately from this.

(5) Let

$$C_X = \Big(\bigcup_{y_j \in T} C_X^j\Big) \cup C_X^0, \qquad C_Y = \Big(\bigcup_{y_j \in T} C_Y^j\Big) \cup C_Y^0 \cup T,$$
$$R_X = V_X - C_X \quad \text{and} \quad \alpha = \bigvee C_X \vee \bigvee C_Y \vee \bigvee R_X.$$

Then (i) $r(\alpha) \leqq |C_X| + |C_Y| + |R_X| - |T| - 1 = |C_Y| + k - |T|$ and (ii) $r(\alpha \wedge \beta) \geqq |C_Y|$.

PROOF. To verify (i), we observe that the set $S = C_0 \cup \bigcup_{Y_j \in T} C_j$ has rank $\leqq |S| - |T| - 1$, by Lemma 3. Since α is obtained by adding the set R_X to S, the inequality follows. Inequality (ii) follows immediately from (3) and the fact that C_Y is independent.

(6) $r(\alpha \vee \beta) \leqq n - 1$.

PROOF. By the submodular inequality, $r(\alpha \vee \beta) \leqq r(\alpha) + r(\beta) - r(\alpha \wedge \beta)$. But $r(\beta) = n - k - 1 + |T|$, and substituting the results of (5) gives the inequality $r(\alpha \vee \beta) \leqq (|C_Y| + k - |T|) + (n - k - 1 + |T|) - |C_Y| = n - 1$.

(7) This is impossible.

PROOF. α contains x_{k+1}, since it contains C_0, and also every $x \in V_X$. β contains every $x \in U_X$ except x_{k+1}, by (4). Hence $\alpha \vee \beta$ contains every $x \in X$, and so must have rank n.

This contradiction shows that some admissible sequence must lead to a "switchable" y, and the proof is complete.

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