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ANOTHER EXCHANGE PROPERTY FOR BASES

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ABSTRACT. Let X and Y be bases of a combinatorial geometry of rank n. If $A \subseteq X$ and $B \subseteq Y$, with $|A| + |B| \geq n+1$, then there exist subsets $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $(X - A_0) \cup B_0$ and $(Y - B_0) \cup A_0$ are both bases.

In this note we prove the following result, originally conjectured by G.-C. Rota.

Theorem. *Let X and Y be bases of a vector space of finite dimension n. If $A \subseteq X$ and $B \subseteq Y$, with $|A| + |B| \geq n+1$, then there exist subsets $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $(X - A_0) \cup B_0$ and $(Y - B_0) \cup A_0$ are both bases.*

It will be clear from the proof that the result holds if X and Y are bases of a combinatorial geometry. It is related to another "exchange" property proved by the author [1], but the proof is much more elementary.

We will use the following notation:

Let $X = \{x_1, \cdots, x_n\}$, $Y = \{y_1, \cdots, y_n\}$, $A = \{x_k, \cdots, x_n\}$, $B = \{y_1, \cdots, y_k\}$. For each $i = 1, \cdots, k$, let C_i be the minimal dependent set (circuit) determined by y_i and X. We also write $A_i = C_i \cap A$, $A_i' = C_i \cap (X - A)$, so that $C_i = y_i \cup A_i \cup A_i'$. For each $S \subseteq B$, let $\alpha_S = \bigwedge_{y \in S} \hat{y}$, where $\hat{y} = \overline{Y - y}$, the linear closure of $Y - y$. (Equivalently, $\alpha_S = \overline{Y - S}$.) Finally, we let $F = \{S \subseteq B \mid A_i \not\subseteq \alpha_S \text{ for all } y_i \in S\}$.

Lemma 1. *F is nonempty.*

Proof. Suppose not. Then, relabelling the y_i's if necessary, we may suppose that

$$A_1 \subseteq \alpha_B, \quad A_2 \subseteq \alpha_B \vee y_1, \quad A_3 \subseteq \alpha_B \vee y_1 \vee y_2, \cdots,$$
$$A_k \leq \alpha_B \vee y_1 \vee \cdots \vee y_{k-1}.$$

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(Choose y_1 such that $A_1 \leq \alpha_B$; remove it, and continue, etc.) But then, if $\beta = \alpha_B \vee x_1 \vee \cdots \vee x_{k-1}$, we have

$$\beta \geq A_1 \quad \text{and} \quad A_1', \quad \text{and hence } y_1,$$

$$\beta \geq A_2 \quad \text{and} \quad A_2', \quad \text{and hence } y_2,$$

$$\beta \geq A_k \quad \text{and} \quad A_k', \quad \text{and hence } y_k.$$

This implies that $\beta \geq y_1, y_2, \cdots, y_k$ and α_B, which is impossible, since $r(\beta) \leq n-1$.

Lemma 2. *Any minimal member S of F is exchangeable.*

Proof. We may suppose that $|S| > 1$, since otherwise the conclusion follows trivially. If $y_i \in S$, then, since $S - y_i \notin F$, there exists $y_j \in S - y_i$ such that $A_j \subseteq \alpha_{S-y_i} = \alpha_S \vee y_i$. If we write $j = f(i)$, then it is clear that $f: S \to S$ is a bijection. (Since $f(i) = f(i') = j \Rightarrow A_j \leq (\alpha_S \vee y_i) \wedge (\alpha_S \vee y_i') = \alpha_S$, contradicting the fact that $S \in F$.) For each $y_i \in S$, choose an element $z_i \in A_{f(i)} - \alpha_S$. Since $\alpha_S \vee y_i = \alpha_S \vee z_i$ for each $y_i \in S$, it is clear that S may be replaced by $\{z_i\}_{y_i \in S}$. Moreover, z_i does not appear in A_j for any $j \neq f(i)$, since the sets $A_j - \alpha_S$ are disjoint. Hence, the z_i's may be exchanged successively for the y_i's. (That is, successively adding the y_i's to X and removing the z_i's does not alter any of the remaining circuits, and so the same elements are exchangeable.) This completes the proof.

REFERENCE

1. C. Greene, *A multiple exchange property for bases*, Proc. Amer. Math. Soc. 39 (1973), 45–50.

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