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Realizable Hamiltonians for universal adiabatic quantum computers

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It has been established that local lattice spin Hamiltonians can be used for universal adiabatic quantum computation. However, the two-local model Hamiltonians used in these proofs are general and hence do not limit the types of interactions required between spins. To address this concern, the present paper provides two simple model Hamiltonians that are of practical interest to experimentalists working toward the realization of a universal adiabatic quantum computer. The model Hamiltonians presented are the simplest known quantum-Merlin-Arthur-complete (QMA-complete) two-local Hamiltonians. The two-local Ising model with one-local transverse field which has been realized using an array of technologies, is perhaps the simplest quantum spin model but is unlikely to be universal for adiabatic quantum computation. We demonstrate that this model can be rendered universal and QMA-complete by adding a tunable two-local transverse $\sigma^x\sigma^x$ coupling. We also show the universality and QMA-completeness of spin models with only one-local $\sigma^x$ and $\sigma^z$ fields and two-local $\sigma^x\sigma^z$ interactions.

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I. INTRODUCTION

What are the minimal physical resources required for universal quantum computation? This question is of interest in understanding the connections between physical and computational complexity, and for any practical implementation of quantum computation. In 1982, Barahona [1] showed that finding the ground state of the random field Ising model is NP-hard. Such observations fostered approaches to solving problems based on classical [2] and later quantum annealing [3]. The idea of using the ground-state properties of a quantum system for computation found its full expression in the adiabatic model of quantum computation [4]. This model works by evolving a system from the accessible ground state of an initial Hamiltonian $H_i$ to the ground state of a final Hamiltonian $H_f$, which encodes a problem’s solution. The evolution takes place over parameters $s \in [0,1]$ as $H(s) = (1-s)H_i + sH_f$, where $s$ changes slowly enough that transitions out of the ground state are suppressed [5]. The simplest adiabatic algorithms can be realized by adding noncommuting transverse field terms to the Ising Hamiltonian $\sum_i h_i \sigma^z_i + \sum_i \Delta_i \sigma^y_i + \sum_i J_i \sigma^x_i \sigma^x_j$, (see Ref. [6]). However, it is unlikely that the Ising model with transverse field can be used to construct a universal adiabatic quantum computer [7].

What then are the simplest Hamiltonians that allow universal adiabatic quantum computation? For this we turn to the complexity class quantum-Merlin-Arthur (QMA), the quantum analog of NP, and consider the QMA-complete problem $k$-LOCAL HAMILTONIAN[1] [8]. One solves $k$-LOCAL HAMILTONIAN by determining if there exists an eigenstate with energy above a given value or below another—with a promise that one of these situations is the case—when the system has at most $k$-local interactions. A “yes” instance is shown by providing a witness eigenstate with energy below the lowest promised value.

The problem 5-LOCAL HAMILTONIAN was shown to be QMA-complete by Kitaev [8]. To accomplish this, Kitaev modified the autonomous quantum computer proposed by Feynman [9]. This modification later inspired a proof of the polynomial equivalence between quantum circuits and adiabatic evolutions by Aharonov et al. [10] (see also Refs. [11,12]). Kempe, Kitaev, and Regev subsequently proved QMA-completeness of 2-LOCAL HAMILTONIAN [14]. Oliveira and Terhal then showed that universality remains even when the two-local Hamiltonians act on particles in a subgraph of the two-dimensional (2D) square lattice [15]. Any QMA-complete Hamiltonian may realize universal adiabatic quantum computation, and so these results are also of interest for the implementation of quantum computation.

Since the 1-LOCAL HAMILTONIAN is efficiently solvable, an open question is to determine which combinations of two-local interactions allow one to build QMA-complete Hamiltonians. Furthermore, the problem of finding the minimum set of interactions required to build a universal adiabatic quantum computer is of practical, as well as theoretical, interest: every type of two-local interaction requires a separate quantum computer is of practical, as well as theoretical, interest: every type of two-local interaction requires a separate type of physical interaction. To address this question we prove the following theorems:

Theorem 1. The problem: the 2-LOCAL ZZXX HAMILTONIAN is QMA-complete, with the ZZXX Hamiltonian given as

$$H_{ZZXX} = \sum_i h_i \sigma^z_i + \sum_i \Delta_i \sigma^y_i + \sum_{i<j} J_{ij} \sigma^x_i \sigma^x_j + \sum_{i<j} K_{ij} \sigma^z_i \sigma^z_j. \tag{1}$$

Theorem 2. The problem: 2-LOCAL ZX HAMILTONIAN is QMA-complete, with the ZX Hamiltonian given as

$$H_{ZX} = \sum_i h_i \sigma^x_i + \sum_i \Delta_i \sigma^y_i + \sum_{i<j} J_{ij} \sigma^x_i \sigma^x_j + \sum_{i<j} K_{ij} \sigma^z_i \sigma^z_j. \tag{2}$$

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1In computer science parlance, “problems” such as SAT, and LOCAL HAMILTONIAN are capitalized.

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Structure. In the present paper we briefly review the standard circuit to adiabatic construction to show that 2-LOCAL HAMILTONIAN is QMA-complete when restricted to real-valued Hamiltonians. We then show how to approximate the ground states of such two-local real Hamiltonians by the ZX and ZXXX Hamiltonians. We conclude this work by providing references confirming our claim that the Hamiltonians in Eqs. (1) and (2) are highly relevant to experimentalists attempting to build a universal adiabatic quantum computer.

II. THE PROBLEM

The translation from quantum circuits to adiabatic evolutions began when Kitaev [8] replaced the time dependence of gate model quantum algorithms with spatial degrees of freedom using the nondegenerate ground state of a positive semidefinite Hamiltonian:

\[ 0 = H|\psi_{\text{ini}}\rangle = (H_{\text{in}} + H_{\text{clock}} + H_{\text{classinit}} + H_{\text{prop}})|\psi_{\text{ini}}\rangle. \]  

To describe this, let \( T \) be the number of gates in the quantum circuit with gate sequence \( U_1 \cdots U_2 U_1 \) and let \( n \) be the number of logical qubits acted on by the circuit. Denote the circuit’s classical input by \(|x\rangle\) and its output by \(|\psi_{\text{out}}\rangle\). The history state representing the circuit’s entire time evolution is

\[
|\psi_{\text{hist}}\rangle = \frac{1}{\sqrt{T+1}} \left( |0\rangle \otimes U_T |x\rangle \otimes (|0\rangle \otimes 0)^{T-1} + U_2 U_1 |x\rangle \otimes (|1\rangle \otimes 0)^{T-2} + \cdots + U_2 \cdots U_2 U_1 |x\rangle \otimes |0\rangle^T \right),
\]

where we have indexed distinct time steps by a \( T \) qubit unary clock. In the following, tensor product symbols separate operators acting on logical qubits (left) and clock qubits (right).

\( H_{\text{in}} \) acts on all \( n \) logical qubits and the first clock qubit. By annihilating time-zero clock states coupled with classical input \( x \), \( H_{\text{in}} \) ensures that valid input state \(|x\rangle \otimes |0\rangle \cdots 0\rangle\) is in the low-energy eigenspace:

\[
H_{\text{in}} = \sum_{i=1}^{n} (1 - |x_i\rangle\langle x_i|) \otimes |0\rangle\langle 0| + \left( \frac{1}{4} \right)^n \sum_{i=1}^{n} [1 - (-1)^i \sigma^x_i] \otimes (1 + \sigma^y_i). \]

\( H_{\text{clock}} \) is an operator on clock qubits ensuring that valid unary clock states \(|00\cdots 0\rangle, |10\cdots 0\rangle, |110\cdots 0\rangle\), etc., span the low-energy eigenspace

\[
H_{\text{clock}} = \sum_{i=1}^{T-1} |01\rangle\langle 01|_{(i,i+1)} = \frac{1}{4} \left( (T-1)I + \sigma^x_i - \sigma^x_T - \sum_{r=1}^{T-1} \sigma^x_r \sigma^x_{(r+1)} \right),
\]

where the superscript \((i,i+1)\) indicates the clock qubits acted on by the projection. This Hamiltonian has a simple physical interpretation as a line of ferromagnetically coupled spins with twisted boundary conditions, so that the ground state is spanned by all states with a single domain wall. The term \( H_{\text{classinit}} \) applies a penalty \(|1\rangle\langle 1| \otimes \) to the first qubit to ensure that the clock is in state \(|0\rangle^T(0)\) at time \( t=0 \).

\( H_{\text{prop}} \) acts both on logical and clock qubits. It ensures that the ground state is the history state corresponding to the given circuit. \( H_{\text{prop}} \) is a sum of \( T \) terms, \( H_{\text{prop}} = \sum_{t=1}^{T} H_{\text{prop},t} \), where each term checks that the propagation from time \( t-1 \) to \( t \) is correct. For \( 2 \leq t \leq T-1 \), \( H_{\text{prop},t} \) is defined as

\[
\text{def} \quad H_{\text{prop},t} = 1 \otimes (t-1)(t-1) - U_t \otimes |t\rangle\langle t| - U_t^\dagger \otimes (t-1)\langle t| + 1 \otimes |t\rangle\langle t|,
\]

where operators \(|t\rangle\langle t| = \{110\cdots 0\}_{(t-1,t+1)}\), etc., act on clock qubits \( t-1, t \), and \( t+1 \) and where the operator \( U_t \) is the \( n \)th gate in the circuit. For the boundary cases \((t=1, T)\), one writes \( H_{\text{prop}} \) by omitting a clock qubit \((t=1 \text{ and } t+1, \text{ respectively})\).

We have now explained all the terms in the Hamiltonian from Eq. (3)—a key building block used to prove the QMA-completeness of 5-LOCAL HAMILTONIAN [8]. The construction reviewed in the present section was also used in a proof of the polynomial equivalence between quantum circuits and adiabatic evolutions [10]. Which physical systems can implement the Hamiltonian model of computation from Eq. (3)? Ideally, we wish to find a simple Hamiltonian that is in principle realizable using current, or near-future technology. The ground states of many physical systems are real-valued, such as the ground states of the Hamiltonians from Eqs. (1) and (2). So a logical first step in our program is to show the QMA-completeness of general real-valued local Hamiltonians.

A. The QMA-completeness of real-valued Hamiltonians

Bernstein and Vazirani showed that arbitrary quantum circuits may be represented using real-valued gates operating on real-valued wave functions [16]. Using this idea, one can show that 5-LOCAL REAL HAMILTONIAN is already QMA-complete—leaving the proofs in Ref. [8] otherwise intact and changing only the gates used in the circuits. \( H_{\text{in}} \) from Eq. (5) and \( H_{\text{clock}} \) from Eq. (6) are already real-valued and at most two-local. Now consider the terms in \( H_{\text{prop}} \) from Eq. (7) for the case of self-inverse elementary gates \( U_{t} = U_{t}^\dagger \)

\[
H_{\text{prop},t} = \frac{1}{4} \left( (1 - \sigma^x_{(t-1)})(1 + \sigma^x_{(t+1)}) - U_t \otimes (1 - \sigma^x_{(t-1)})\sigma^x_{(t+1)} \right).
\]

For the boundary cases \((t=1, T)\), we define

\[
H_{\text{prop},1} = \frac{1}{2} (1 + \sigma^x) - U_1 \otimes \sigma^x_{(t+1)} + \frac{1}{2} (\sigma^x + \sigma^x_{(t+1)}),
\]

\[
H_{\text{prop},T} = \frac{1}{2} (1 - \sigma^x_{(t-1)}) - U_T \otimes (\sigma^x - \sigma^x_{(t-1)}).
\]

The terms from Eqs. (8) and (9) acting on the clock space are already real-valued and at most three-local. As an explicit example of the gates \( U_{n} \) let us define a universal real-valued and self-inverse two-qubit gate.
To prove our theorems, we first give the Hamiltonians—the Hamiltonians from Eqs. (1) and (2), respectively. We do this using perturbation theory [14,15] to construct Hamiltonians that approximate the operators \(\sigma_i^x \sigma_j^x\) as well as the operators \(\sigma_i^x \sigma_j^z\) and \(\sigma_i^z \sigma_j^x\) with terms from the ZZXX Hamiltonian.

**B. The ZZXX gadget**

We use the ZZXX Hamiltonian from Eq. (1) to construct the interaction \(\sigma_i^x \sigma_j^x\) from \(\sigma^x \sigma^x\) and \(\sigma^z \sigma^z\) interactions. Let \(H_{\text{eff}} = \alpha \sigma_i^x \sigma_j^x \otimes |0\rangle\langle 0|\), where qubit \(k\) is an ancillary qubit and define the penalty Hamiltonian \(H_p\) and corresponding Green’s function \(G(z)\) as follows:

\[
H_p = \delta |1\rangle \langle 1|_k - \frac{\delta}{2} (1 - \sigma_z^k) \quad \text{and} \quad G(z) = (z - H_p)^{-1}.
\]

\[H_p\] splits the Hilbert space into a degenerate low-energy eigenspace \(\mathcal{L}_- = \text{span}\{|s, \bar{s}angle\langle 0| \mid s, \bar{s} \in \{0, 1\}\}\) with qubit \(k\) is |0\rangle, and a \(\delta\) energy eigenspace \(\mathcal{L}_+ = \text{span}\{|s, \bar{s}angle\langle 1| \mid s, \bar{s} \in \{0, 1\}\}\) with qubit \(k\) is |1\rangle.

First, we give the ZZXX Hamiltonian which produces an effective \(\sigma^x \sigma^x\) interaction in the low-energy subspace. Let \(V\) be an arbitrary ZZXX Hamiltonian acting on qubits \(i\) and \(j\) and consider a perturbation \(V = V_1 + V_2 + V_3\) that breaks the \(\mathcal{L}_-\) zero eigenspace degeneracy by creating an operator \(O(e)\) close to \(H_{\text{eff}}\) in this space:

\[V_1 = \left[ Y + D(\sigma_i^x + 1) \right] \otimes 1_k - A \sigma_i^z \otimes |0\rangle\langle 0|_k,
\]

\[V_2 = B(\sigma_j^x + 1) \otimes \sigma_i^z, \quad V_3 = C \sigma_i^z \otimes |1\rangle\langle 1|_k.\]

The term \(V_2\) above allows the mediator qubit \(k\) to undergo virtual excitations and applies an \(\alpha^x\) term to qubit \(j\) during transitions between the \(\mathcal{L}_-\) and \(\mathcal{L}_+\) subspaces. During excitation into \(\mathcal{L}_+\), the term \(V_3\) applies a \(\alpha^z\) term to qubit \(i\). This perturbation is illustrated in Fig. 1.

Let \(\Pi_{\pm}\) be projectors on \(\mathcal{L}_{\pm}\); for arbitrary operator \(O\) we define \(O_{\pm} = \Pi_{\pm} O \Pi_{\pm}\) and let \(\lambda(O)\) denote the lowest eigenvalue of \(O\). One approximates \(\lambda(H_{\text{tar}})\) of the desired low-energy effective two-local Hamiltonian by a realizable two-local physical Hamiltonian \(\tilde{H} = H + V\), where \(\lambda(H)\) is calculated using perturbation theory. The spectrum of \(\tilde{H}_-\) is approximated by the projection of the self-energy operator \(\Sigma(z)\) for real-valued \(z\) which has the following series expansion:

\[\Sigma_{-}(z) = \tilde{H}_{-}^0 + \tilde{V}_{-}^0 + \tilde{V}_{-}^0 G_{+}(z) \tilde{V}_{-}^0 + \tilde{V}_{-}^0 G_{+}(z) \tilde{V}_{-}^0 G_{+}(z) \tilde{V}_{-}^0 + \cdots.
\]

Note that with our penalty Hamiltonian \(H_{-} = 0\), and for the perturbing Hamiltonian \(V = V_1 + V_2 + V_3\) only \(V_1\) is nonzero in the low energy subspace, \(V_1\) and \(V_3\) are nonzero in the high-energy subspace, and only \(V_2\) induces transitions between the two subspaces. The nonzero projections are:

\[V_{1-} = \left[ Y + A \sigma_i^z + D(\sigma_i^x + 1) \right] \otimes |0\rangle\langle 0|_k.
\]

\[V_{2-} = B(\sigma_j^x + 1) \otimes |0\rangle\langle 0|_k,
\]

\[V_{2+} = B(\sigma_j^x + 1) \otimes |1\rangle\langle 1|_k,
\]

\[V_{3+} = V_3,
\]

\[V_4 = \left[ Y + C \sigma_i^z + D(\sigma_i^x + 1) \right] \otimes |1\rangle\langle 1|_k.
\]

The series expansion of the self-energy follows directly:
express the dressed Hamiltonian

In practice there will always be some interaction between
final physical interaction dressed by the effect of virtual exci-
tations. These terms can be approximated with only
$O(\varepsilon)$, the error terms in Eq. (22) must be bounded above by
$\varepsilon$ through an appropriate choice of $\delta$. Define a lower
bound on the spectral gap $\delta$ as an inverse polynomial in $\varepsilon$:
$\delta \approx E_0^{-r}$, where $E_0$ is a constant and integer $r \geq 1$. Now
bound $r$ by considering the (weak) upper bound on $\|V\|:

$$
\|V\| \leq \|\bar{V}\| + |\alpha_{ij}| + 4\delta^{3/23} + 2E_0\left(\frac{\delta}{E_0}\right)^{2/3} + \frac{1}{2}\left(\frac{\delta}{E_0}\right)^{2/3}.
$$

(23)

The largest term in $\delta^{-3}\|\bar{V}\|$ is $O(\bar{E_0}/\delta^{1/3})$, and so in order that
$\delta^{-3}\|\bar{V}\| < \varepsilon$ we require $r \geq 3$. This also bounds the term
below fourth order, $\bar{E}_0^{4/3}\delta^{-1/3} = O(\bar{E}_0)$ and so for $\varepsilon \ll \delta$ we obtain
$\|\Sigma_-(z) - H_{\text{eff}}\| = O(\varepsilon)$. In fact, $\Sigma_-(0) = H_{\text{eff}} + \bar{E}_0 \sigma_z$.
Now apply theorem (3) from Ref. [14] and it follows that
$\lambda(\tilde{H}_0) - \lambda(\tilde{H}) = O(\varepsilon)$. It also follows from lemma (11) of
Ref. [14] that the ground-state wave function of $H_{\text{eff}}$ is also
close to the ground state of our gadget.

The $ZZXX$ Hamiltonian (1) allows for the direct realiza-
tion of all terms in Eq. (10) except for $\sigma^x \sigma^x$ and $\sigma^y \sigma^y$ inter-
actions. These terms can be approximated with only $O(\varepsilon)$ error using the gadget in the present section—thereby showing
that the $ZZXX$ Hamiltonian can efficiently approximate all terms from Eq. (10). Similarly, the $ZX$ Hamiltonian allows for
the direct realization of all terms in Eq. (10) except for $\sigma^x \sigma^x$ and $\sigma^y \sigma^y$ interactions. These terms will be approxi-
mated with only $O(\varepsilon)$ error by defining gadgets in the coming
sections—showing that the $ZX$ Hamiltonian can also be
used to efficiently approximate all terms from Eq. (10).

C. The $ZZ$ from $ZX$ gadget

We approximate the operator $\beta_{ij} \sigma_i \sigma_j$ using the $ZX$ Hamil-
tonian in Eq. (2) by defining a penalty Hamiltonian as in Eq.
(11). The required perturbation is a sum of terms $V = V_1 + V_2$:

$$
V_1 = Y + A|0\rangle\langle 0|_k, \quad V_2 = B(\sigma_i^x - \sigma_j^x) \otimes \sigma_k^x.
$$

(24)
The nonzero projections are
\[ V_{1+} = Y \otimes |1\rangle\langle 1|_i, \quad V_{1-} = (Y + A_1) \otimes |0\rangle\langle 0|_k, \]
\[ V_{2+} = B(\sigma_i^+ - \sigma_j^+ \otimes |1\rangle\langle 0|_k, \quad V_{2-} = B(\sigma_i^- - \sigma_j^- \otimes |0\rangle\langle 1|_k. \]
\[ (25) \]

\( V_1 \) does not couple the low- and high-energy subspaces and \( V_2 \) couples the subspaces but is zero in each subspace. The series expansion of the self-energy follows directly:

1st: \[ (Y + A_1) \otimes |0\rangle\langle 0|_k, \]
2nd: \[ \frac{B^2}{2} (\sigma_j^\dagger - \sigma_j^\dagger)^2 (\sigma_i^\dagger - \sigma_i^\dagger) \otimes |0\rangle\langle 0|_k, \]
3rd: \[ \frac{B^2}{2} (\sigma_j^\dagger - \sigma_j^\dagger) Y(\sigma_i^\dagger - \sigma_i^\dagger) \otimes |0\rangle\langle 0|_k. \] (26)

Note that in this case the desired terms appear at second order in the expansion, rather than at third order as was the case for the ZX from ZZXX gadget. The terms which dress the physical Hamiltonian \( Y \) coupling qubits \( i \) and \( j \) appear at third order. The series expansion of the self-energy in the low-energy subspace is
\[ \Sigma(z) = \frac{2B^2 (1 - \sigma_j^\dagger \sigma_j^\dagger)}{(z - \delta^2)} + O(|V|^4 \delta^3), \] (27)
where the dressed interaction \( Y \) is defined as
\[ Y = Y + \frac{B^2}{2} (\sigma_j^\dagger - \sigma_j^\dagger) Y(\sigma_j^\dagger - \sigma_j^\dagger). \] (28)

We assume that the physical interaction \( Y \) between \( i \) and \( j \) qubits is a ZX Hamiltonian and express the dressed Hamiltonian in terms of modified coupling constants. Writing the physical Hamiltonian
\[ Y = h_i \sigma_i^\dagger + h_j \sigma_j^\dagger + \Delta_i \sigma_i^\dagger + \Delta_j \sigma_j^\dagger + J_{ij} \sigma_i^\dagger \sigma_j^\dagger + K_{ij} \sigma_i^\dagger \sigma_j^\dagger, \] (29)

We obtain modified coupling strengths
\[ h_i \rightarrow h_i + \frac{2B^2 (h_i - h_j)}{(z - \delta^2)}, \quad h_j \rightarrow h_j + \frac{2B^2 (h_j - h_i)}{(z - \delta^2)}. \] (30)

In this case only the local \( Z \) field strengths are modified.

We choose values for the perturbation interaction strengths as follows: \( B = \sqrt{\frac{\epsilon}{\alpha^2}} \) and \( \Delta = \beta_i \) and expand the self-energy in the limit where \( z \) is constant \([z = 0(1) \ll \delta]\):
\[ \Sigma(0) = \tilde{Y} + \beta_i \sigma_i^\dagger \sigma_j^\dagger + O(|V|^4 \delta^3). \] (31)

We again choose \( \delta \) to be an inverse power in a small parameter \( \epsilon \) so that \( \delta = \tilde{E} \epsilon^{-r} \), and again use the (weak) upper bound on \( |V|\):
\[ |V| \leq |\tilde{Y}| + \beta_i + \sqrt{2} \beta_j \delta. \] (32)

The largest term in \( |V|^4 \delta^3 \) is \( 4 \beta_i^2 \delta^{-1} \), and so in order that \( |V|^4 \delta^3 < \epsilon \) we require \( r \geq 1 \).

Using the gadget defined in the present section, the ZX Hamiltonian can now be used to efficiently approximate all terms in Eq. (10) except for \( \sigma_i^\dagger \sigma_j^\dagger \) interactions. These interactions can also be approximated with only \( O(\epsilon) \) error by defining an additional gadget in the next section.

**D. The XX from ZX gadget**

An \( \sigma_i^\dagger \sigma_j^\dagger \) coupling may be produced from the \( \sigma_i^\dagger \sigma_j^\dagger \) coupling as follows. We define a penalty Hamiltonian and corresponding Green’s function
\[ H_p = \frac{\delta}{2} (1 - \sigma_i^\dagger \sigma_i^\dagger) = \delta \langle \cdot | - | \cdot \rangle, \quad G_{++} = \frac{1}{z - \delta} \langle \cdot | - | \cdot \rangle \] (33)

This penalty Hamiltonian splits the Hilbert space into a low-energy subspace in which the ancilla qubit \( k \) is in state \( |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2} \) and a high-energy subspace in which the ancilla qubit \( k \) is in state \( |-\rangle = (|0\rangle - |1\rangle)/\sqrt{2} \).

The perturbation is a sum of two terms \( V = V_1 + V_2 \), where \( V_1 \) and \( V_2 \) are given by
\[ V_1 = Y \otimes |1\rangle\langle 1|_k + A + |\rangle\langle +|_k, \quad V_2 = B(\sigma_i^\dagger - \sigma_j^\dagger) \sigma_j^\dagger. \] (34)

The nonzero projections are
\[ V_{1+} = |\rangle\langle -|_i \otimes |\rangle\langle 1|_k, \quad V_{1-} = Y \otimes |\rangle\langle +|_k, \]
\[ V_{2+} = -\langle \rangle\langle -|_k, \quad V_{2-} = B(\sigma_i^\dagger - \sigma_j^\dagger) \sigma_j^\dagger \langle \rangle\langle +|_k. \] (35)

Once more we see that the perturbation \( V_1 \) does not couple the subspaces, whereas \( V_2 \) couples the subspaces but is zero in each subspace. This perturbation is illustrated in Fig. 3.

The series expansion of the self-energy follows:

1st: \[ (Y + A_1) \otimes |\rangle\langle +|_k, \]
2nd: \[ \frac{B^2 (\sigma_j^\dagger - \sigma_j^\dagger)^2}{(z - \delta^2)} \otimes |\rangle\langle +|_k, \]
3rd: \[ \frac{B^2}{2} (\sigma_j^\dagger - \sigma_j^\dagger) Y(\sigma_j^\dagger - \sigma_j^\dagger). \] (36)

Again we see that the desired term appears at second order, while the third-order term is due to the dressing of the physical interaction \( Y \) between qubits \( i \) and \( j \). In the low-energy subspace the series expansion of the self-energy to third order is
\[ \Sigma(z) = \tilde{Y} + A_1 + \frac{2B^2 (1 - \sigma_j^\dagger \sigma_j^\dagger)}{(z - \delta^2)} + O(|V|^4 \delta^3), \] (37)
where the dressed interaction \( \tilde{Y} \) is defined as
As before, this self-energy may be made

\[
\frac{1}{A/2}.
\]

which allowed us to approximate [with \(O(\varepsilon)\) error] all the Hamiltonian terms from Eq. (3) using either the \(ZZXX\) or \(ZX\) Hamiltonians. It also follows from Lemma (11) of Ref. [14] that the ground-state wave function of \(H_{\text{eff}}\) is also close to the ground state of our gadget. So to complete our proof, it is enough to show that each gadget satisfies the criteria given in theorem (3) from Ref. [14].

\[\Sigma(0)_- = \tilde{Y} \otimes (+|_k + \gamma_j \sigma^j \sigma^j_\alpha \otimes +|_k + O(|V|^4 \delta^3)).\]

As before, this self-energy may be made \(O(\varepsilon)\) close to the target Hamiltonian by a bound \(\delta \approx E^{-1}\).

\[\Delta_j \rightarrow \Delta_j + \frac{2B^2(\Delta_j - \Delta_j)}{(z - \delta)^2}, \quad \Delta_j \rightarrow \Delta_j + \frac{2B^2(\Delta_j - \Delta_j)}{(z - \delta)^2},\]

III. CONCLUSION

The objective of this work was to provide simple model Hamiltonians that are of practical interest for experimentalists working toward the realization of a universal adiabatic quantum computer. Accomplishing such a task also enabled us to find the simplest known QMA-complete two-local Hamiltonians. The \(\sigma^x \sigma^x\) coupler is realizable using systems including capacitive coupling of flux qubits [18] and spin models implemented with polar molecules [19]. In addition, a \(\sigma^x \sigma^y\) coupler for flux qubits is given in Ref. [10], autonomous [22], measurement-based [23], and universal adiabatic quantum computation [10,14,15], and may also be useful for quantum annealing [24]. For these reasons, the reported Hamiltonians are of interest to those concerned with the practical construction of a universal adiabatic quantum computer.

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[25] The QMA completeness of this subset of Hamiltonians was found independently by D. Bacon (unpublished).