General Refraction problem with Phase Discontinuities on nonflat metasurfaces

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GENERAL REFRACTION PROBLEMS WITH PHASE DISCONTINUITY

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Abstract. This paper provides a mathematical approach to study metasurfaces in non flat geometries. Analytical conditions between the curvature of the surface and the set of refracted directions are introduced to guarantee the existence of phase discontinuities. The approach contains both the near and far field cases. A starting point is the formulation of a vector Snell law in presence of abrupt discontinuities on the interfaces.

Contents

1. Introduction

For classical lens design, a typical problem is to find two surfaces so that the region sandwiched between them and filled with an homogeneous material refracts light in a desired manner. For metalens design, a surface is given and the question is to find a function on the surface (a phase discontinuity) so that the pair, surface together with the phase discontinuity (the metalens) refracts light in a desired manner. The subject of metalenses is a flourishing area of research and one of

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the nine runners-up for Science's Breakthrough of the Year 2016 [sci16]. Metalenses have been designed for flat geometries with the scalar generalized laws of reflection and refraction with phase discontinuities, see [YGK+11], [AGY+12], [AKG+12], and the comprehensive review article [YC14]. These general laws have been experimentally observed by using arrays of optical antennas on silicon. The review in [CTY16] describes the past 15 years of progress on metasurfaces, from experimental realization of the generalized laws of refraction, to applications in wavefront and beam shaping. Recently, it has been proven [ARW+16] that at certain frequencies, a thin layer of nanoparticles on a perfectly conducting sheet acts as a metasurface. For more recent work in the area and an extensive up to date bibliography, we refer to [GCA+17]; see also [KZRC+16] and [KCD+16].

The purpose of this paper is to provide a mathematically rigorous foundation to deal with general metasurfaces and to determine the relationships between the curvature of the surface and the phase discontinuity. A problem we solve is the following: when light emanates from a point source, find a metalens that refracts light into a prescribed set of directions or points, see Figure 1. In fact, given a surface and a compatible set of directions, satisfying appropriate curvature type conditions, we show that a phase discontinuity exists on the surface so that the metalens refracts light into the prescribed set of directions, Section 5. Vice versa, given a phase discontinuity and a fixed direction, we find the admissible surfaces for that phase discontinuity and direction, Section 6. Of great importance to answer these questions in general geometries is the formulation of a generalized Snell's law in vector form, Equation (3.3), which is deduced using wave fronts in Section 3. In term of wave vectors, a vector law is formulated in [AKG+12, Equation (2)]. However, Equation (3.3) is effective and flexible for the actual calculation of phase discontinuities in general and to obtain our results. We illustrate these with explicit constructions for planar and spherical interfaces, Sections 4.1, 4.2, 7.1, and 7.2; see also Remark 5.3.

The outline of the paper is as follows. In Section 2, we briefly recall the classical Snell's law for surfaces without phase discontinuities. Then in Section 3 we derive a generalized Snell's law in the presence of a phase discontinuity using wavefronts, Equation (3.3), and analyze the possible critical angles. The far field problem is studied in Section 4 for the plane and the sphere. In Section 5 we allow for variable directions m in the far field. In Section 6, conditions are derived so that given a phase discontinuity a surface exists. Finally, in Section 7 the near field problem is addressed.

2. Background

We recall the classical Snell's law in vector form here. Suppose Γ is a surface in $\mathbb{R}^3$ that separates two media I and II that are homogeneous and isotropic, with

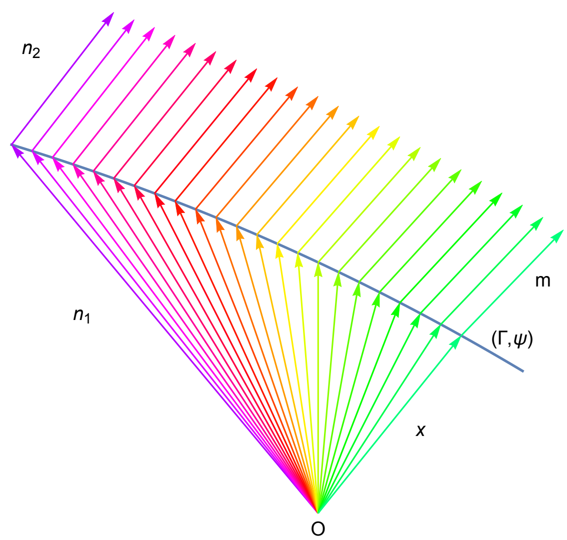

FIGURE 1. Metalens refracting into a fixed direction

refractive indices n_1 and n_2 respectively. If a ray of light* having direction $x \in S^2$, the unit sphere in $\mathbb{R}^3$, and traveling through medium I strikes Γ at the point P, then this ray is refracted in the direction $m \in S^2$ through medium II according to the Snell law in vector form:

$$n_1(x \times \nu) = n_2(m \times \nu), \tag{2.1}$$

where ν is the unit normal to the surface to Γ at P pointing towards medium II; see [Lun64, Subsection 4.1]. It is assumed here that $x \cdot \nu \geq 0$.

This has several consequences:

(a) the vectors x, m, ν are all on the same plane (called the plane of incidence);
(b) the well known Snell's law in scalar form holds:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

where θ_1 is the angle between x and ν (the angle of incidence), and θ_2 is the angle between m and ν (the angle of refraction).

Equation (2.1) is equivalent to $(n_1 x - n_2 m) \times \nu = 0$, which means that the vector $n_1 x - n_2 m$ is parallel to the normal vector ν. If we set $\kappa = n_2/n_1$, then

$$x - \kappa\, m = \lambda \nu, \tag{2.2}$$

for some $\lambda \in \mathbb{R}$. Notice that (2.2) univocally determines λ. Taking dot products with x and m in (2.2) we get $\lambda = \cos\theta_1 - \kappa\cos\theta_2$, $\cos\theta_1 = x \cdot \nu > 0$, and $\cos\theta_2 =$

*Since the refraction angle depends on the frequency of the radiation, we assume that light rays are monochromatic.

$m \cdot \nu = \sqrt{1-\kappa^{-2}[1-(x\cdot\nu)^2]}$. In fact, there holds

$$\lambda = x\cdot\nu - \kappa\,\sqrt{1-\kappa^{-2}\,(1-(x\cdot\nu)^2)}. \tag{2.3}$$

The formulation (2.2) is useful solve refraction problems for lens design, see [GH09], [GM13], [GS14], [GT13], and [DLGM17] for a numerical implementation.

3. Derivation of a Vector Snell Law with phase discontinuity using wavefronts

Let n_1, n_2 be the refractive indices of two homogeneous media I and II, respectively. Suppose a surface Γ separates the two media, and an incoming light ray in medium I with wave vector $\mathbf{k}_1$ strikes Γ. Assume that there is a real-valued function ψ, *the phase discontinuity*, defined in a neighborhood of the surface Γ. Notice that ψ must be defined in a neighborhood of Γ because the gradient of ψ will be considered. If ν denotes the unit normal vector to Γ, then the refracted wave vector $\mathbf{k}_2$ satisfies [AKG+12, Equation (2)]:

$$\nu\times(\mathbf{k}_2-\mathbf{k}_1) = \nu\times\nabla\psi \tag{3.1}$$

We give an alternate formulation and derivation of this result by using wavefronts; our starting point is [Gut14, Section 2.2]. For each t, $\Psi(x,y,z,t)=0$ denotes a surface in the variables x,y,z that separates the part of the space that is at rest from the part of the space that is disturbed by the electric and magnetic fields. This surface is called a *wave front*, and the light rays are the orthogonal trajectories to the wave fronts at each time t. We assume that $\Psi_t \neq 0$, and so we can solve $\Psi(x,y,z,t)=0$ in t, obtaining that $\phi(x,y,z)=ct$; so letting t run, the wave fronts are then the level sets of $\phi(x,y,z)$.

Let n_1, n_2, and Γ be as above. An incoming wave front Ψ_1 on medium I strikes the surface Γ and it is then transmitted into a wave front Ψ_2 in medium II (of course, there is also a wave front reflected back). Assuming as before that $(\Psi_j)_t \neq 0$, $j=1,2$, and solving in t, we get that the wave fronts are given by $\phi_j(x,y,z)=ct$ for $j=1,2$, respectively. Suppose the surface Γ is parameterized by $x=f(\xi,\eta)$, $y=g(\xi,\eta)$, $z=h(\xi,\eta)$. If there were no phase discontinuity on the surface Γ, then we would have $\phi_1=\phi_2$ along Γ. But since there is now a phase discontinuity ψ on Γ, we have the following jump condition along Γ:

$$\phi_1(f(\xi,\eta),g(\xi,\eta),h(\xi,\eta)) - \phi_2(f(\xi,\eta),g(\xi,\eta),h(\xi,\eta)) = \psi(f(\xi,\eta),g(\xi,\eta),h(\xi,\eta)).$$

Taking derivatives in ξ and η yields

$$\left(\nabla\phi_1-\nabla\phi_2-\nabla\psi\right)\cdot(f_\xi,g_\xi,h_\xi)=0,$$

and

$$\left(\nabla\phi_1-\nabla\phi_2-\nabla\psi\right)\cdot(f_\eta,g_\eta,h_\eta)=0.$$

That is, the vector $\nabla\phi_1-\nabla\phi_2-\nabla\psi$ must be normal to Γ; as such there exists a real number λ such that

$$\nabla\phi_1-\nabla\phi_2-\nabla\psi=\lambda\nu \tag{3.2}$$

where ν is the unit normal to Γ.

Let $\gamma_j(t)$ denote the light rays in medium j having speed v_j, for $j = 1, 2$; i.e., the orthogonal trajectories to ϕ_j. In particular, we have that $\phi_j(\gamma_j(t)) = ct$, and by the chain rule

$$\nabla\phi_j(\gamma_j(t)) \cdot \gamma_j'(t) = c, \quad j = 1, 2$$

If we parameterize the rays so that $|\gamma_j'(t)| = v_j$, then we obtain

$$|\nabla\phi_j(\gamma_j(t))| = \frac{c}{v_j} = n_j, \quad j = 1, 2$$

since $\nabla\phi_j$ is parallel to γ_j'. Letting

$$x = \frac{\nabla\phi_1(\gamma_1(t))}{|\nabla\phi_1(\gamma_1(t))|}, \quad m = \frac{\nabla\phi_2(\gamma_2(t))}{|\nabla\phi_2(\gamma_2(t))|}$$

we obtain from (3.2) the following formula

$$n_1 x - n_2 m = \lambda\nu + \nabla\psi. \tag{3.3}$$

Taking cross products with the unit normal ν in (3.3), we obtain the equivalent formula

$$\nu \times (n_1 x - n_2 m) = \nu \times \nabla\psi. \tag{3.4}$$

Recall that x is the unit direction of the incident ray, m is the unit direction of the refracted ray, ν is the unit outer normal at the incident point on Γ and $\nabla\psi$ is calculated at the incident point. Note that in the case ψ is constant, we recover the classical Snell's law in vector form (2.1)†.

Starting from (3.3), we now calculate λ. Taking dot products in (3.3) and solving for $x \cdot m$ yields

$$x \cdot m = \frac{n_1 - \lambda\, x \cdot \nu - x \cdot \nabla\psi}{n_2}.$$

Next taking dot products in (3.3) with itself, expanding, and substituting $x \cdot m$ from the previous expression, yields that λ satisfies the quadratic equation:

$$\lambda^2 - [2(n_1 x - \nabla\psi) \cdot \nu]\,\lambda + |n_1 x - \nabla\psi|^2 - n_2^2 = 0. \tag{3.5}$$

Solving for λ yields

$$\lambda = (n_1 x - \nabla\psi) \cdot \nu \pm \sqrt{n_2^2 - \left(|n_1 x - \nabla\psi|^2 - [(n_1 x - \nabla\psi) \cdot \nu]^2\right)}. \tag{3.6}$$

†Notice that if ψ =constant, then $n_1\, \nu \times x = n_2\, \nu \times m$. Taking dot product with m yields $n_1\, m \cdot (\nu \times x) = 0$. This means that m is on the plane through the origin having normal $\nu \times x$ which is the plane generated by ν and x. Therefore ν, x, m are all on the same plane, i.e., the plane of incidence. On the other hand, if ψ is not necessarily constant, then from (3.4) $n_1\, \nu \times x = n_2\, \nu \times m + \nu \times \nabla\psi$. Again taking dot product with m yields $n_1\, m \cdot (\nu \times x) = m \cdot (\nu \times \nabla\psi)$, that is, $m \cdot (\nu \times (n_1\, x - \nabla\psi)) = 0$. That is, now the refracted vector m lies on the plane through the origin and perpendicular to the vector $\nu \times (n_1\, x - \nabla\psi)$ where $\nabla\psi$ is calculated at the point on the surface Γ where the ray with direction x strikes it. This shows that in the general case the refracted vector m is not on the plane generated by ν and x.

Since λ must be a real number, the quantity under the square root must be nonnegative, i.e.,

$$n_2^2 \geq |n_1 x - \nabla\psi|^2 - \left[(n_1 x - \nabla\psi)\cdot\nu\right]^2. \tag{3.7}$$

Assuming this for now, it remains to check which sign ($\pm$) to take in (3.6). Dotting (3.3) with ν and using (3.6) yields

$$n_1 x\cdot\nu - n_2 m\cdot\nu = (n_1 x - \nabla\psi)\cdot\nu \pm \sqrt{n_2^2 - \left(|n_1 x - \nabla\psi|^2 - \left[(n_1 x - \nabla\psi)\cdot\nu\right]^2\right)} + \nabla\psi\cdot\nu,$$

so

$$-n_2 m\cdot\nu = \pm\sqrt{n_2^2 - \left(|n_1 x - \nabla\psi|^2 - \left[(n_1 x - \nabla\psi)\cdot\nu\right]^2\right)}.$$

Since $n_2 > 0$ and $m\cdot\nu \geq 0$, we obtain that

$$\lambda = (n_1 x - \nabla\psi)\cdot\nu - \sqrt{n_2^2 - \left(|n_1 x - \nabla\psi|^2 - \left[(n_1 x - \nabla\psi)\cdot\nu\right]^2\right)}. \tag{3.8}$$

We next analyze (3.7), which will yield the critical angles. Equation (3.7) is equivalent to

$$\left(\left(x - \frac{\nabla\psi}{n_1}\right)\cdot\nu\right)^2 \geq \left|x - \frac{\nabla\psi}{n_1}\right|^2 - \kappa^2.$$

Thus, if x is such that

$$\left|x - \frac{\nabla\psi}{n_1}\right| \leq \kappa,$$

then (3.7) holds. On the other hand, if

$$\left|x - \frac{\nabla\psi}{n_1}\right| > \kappa$$

then (3.7) holds when either

$$x\cdot\nu \geq \frac{\nabla\psi}{n_1}\cdot\nu + \sqrt{\left|x - \frac{\nabla\psi}{n_1}\right|^2 - \kappa^2} \quad \text{or} \quad x\cdot\nu \leq \frac{\nabla\psi}{n_1}\cdot\nu - \sqrt{\left|x - \frac{\nabla\psi}{n_1}\right|^2 - \kappa^2}.$$

Therefore, the critical angles between x and ν are θ_c with

$$x\cdot\nu = \cos\theta_c = \frac{\nabla\psi}{n_1}\cdot\nu + \sqrt{\left|x - \frac{\nabla\psi}{n_1}\right|^2 - \kappa^2}$$

or

$$x\cdot\nu = \cos\theta_c = \frac{\nabla\psi}{n_1}\cdot\nu - \sqrt{\left|x - \frac{\nabla\psi}{n_1}\right|^2 - \kappa^2}.$$

Remark 3.1. In two dimensions the critical angles are considered in [YGK+11]. It is assumed there that the interface Γ is the x-axis, the region $y > 0$ is filled with a material with refractive index n_1, and the region $y < 0$ with a material with refractive index n_2. Also the phase discontinuity satisfies that $\nabla\psi$ is constant and is tangential to the interface, i.e., $\nabla\psi = (a, 0)$ with, for example, $a > 0$. Therefore, the above calculations applied to this case yield

$$\cos\theta_c = x\cdot\nu = \sqrt{\left|x - \frac{\nabla\psi}{n_1}\right|^2 - \kappa^2} = \sqrt{1 - \frac{2|\nabla\psi|}{n_1}\cos(\pi/2-\theta_c) + \frac{|\nabla\psi|^2}{n_1^2} - \kappa^2},$$

where $\kappa = \dfrac{n_2}{n_1}$. Squaring both sides we obtain

$$\cos^2\theta_c = 1 - \frac{2|\nabla\psi|}{n_1}\sin\theta_c + \frac{|\nabla\psi|^2}{n_1^2} - \kappa^2,$$

and the critical angles θ_c are therefore the solutions to the equation

$$\sin^2\theta_c - \frac{2|\nabla\psi|}{n_1}\sin\theta_c + \frac{|\nabla\psi|^2}{n_1^2} - \kappa^2 = 0,$$

i.e.,

$$\theta_c = \arcsin\left(\frac{|\nabla\psi|}{n_1} \pm \kappa\right),$$

which is in agreement with [YGK+11, Formula (3)].

In three dimensions the critical angles are considered in [AKG+12]. The interface Γ is the xy-plane, the region $z > 0$ is filled with a material with refractive index n_1, and the region $z < 0$ with a material with refractive index n_2. Also the phase discontinuity is tangential to the interface, i.e., $\nabla\psi = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, 0\right)$ and without loss of generality we may assume $x = (0, y, z)$. Once again, the above calculations applied to this case yield

$$\cos\theta_c = x\cdot\nu = \sqrt{\left|x - \frac{\nabla\psi}{n_1}\right|^2 - \kappa^2} = \sqrt{1 - \frac{2}{n_1}\left|\frac{\partial\psi}{\partial y}\right|\cos(\pi/2-\theta_c) + \frac{|\nabla\psi|^2}{n_1^2} - \kappa^2}.$$

Proceeding as before we find

$$\theta_c = \arcsin\left(\frac{1}{n_1}\frac{\partial\psi}{\partial y} \pm \sqrt{\kappa^2 - \frac{1}{n_1^2}\left|\frac{\partial\psi}{\partial x}\right|^2}\right),$$

recovering [AKG+12, Formula (8)].

Remark 3.2. The reflection case is when $n_1 = n_2$, so (3.3) and (3.8) become

$$x - m = \frac{1}{n_1}\lambda\,\nu + \frac{\nabla\psi}{n_1}, \qquad \lambda = (n_1 x - \nabla\psi)\cdot\nu + \sqrt{n_1^2 - \left(|n_1 x - \nabla\psi|^2 - \left[(n_1 x - \nabla\psi)\cdot\nu\right]^2\right)},$$

with x the unit incident direction, m the unit reflected vector, ν the unit normal to the interface at the striking point, and $\nabla\psi$ at the striking point. Notice that the choice of the plus sign in front of the square root is because for reflection $m\cdot\nu \le 0$.

4. Far field uniformly refracting planar and spherical metalenses

Let Γ be a surface in three dimensional space and V be a vector valued function defined on Γ; $V : \Gamma \to \mathbb{R}^3$. If x is an incident unit direction striking Γ at a point P, and m is the unit refracted direction, then we obtain, dividing by n_1 in the generalized Snell law (3.3), that

$$x - \kappa\, m = \lambda\, \nu(P) + V(P) \tag{4.1}$$

where $\nu(P)$ is the unit outer normal to Γ at P for some $\lambda \in \mathbb{R}$; $\kappa = n_2/n_1$.

Suppose rays emanate from the origin and we are given a fixed unit vector m. Our goal is to answer the following two questions. First, given a surface Γ separating media n_1 and n_2, find a field V defined on Γ so that all rays from the origin are refracted into the direction m. The second question is, given a field V defined in a region of $\mathbb{R}^3$, find a separation surface Γ between n_1 and n_2 within that region so that all rays emanating from the origin are refracted into the direction m.

We begin in this section answering the first question when Γ is either a plane or a sphere, surfaces of traditional interest in optics, showing explicit phase discontinuities. For general surfaces, the first question is considered in Section 5, even for the more general case of variable m. The second question is answered in Section 6.

4.1. **Case of the plane.** Let Γ be the plane $x_1 = a$ in $\mathbb{R}^3$ with $a > 0$. We want to determine a field $V = (V_1, V_2, V_3)$ defined on Γ so that all rays emanating from the origin are refracted into the unit direction $m = (m_1, m_2, m_3)$, with $m_1 > 0$, Figure 2(a).

Using spherical coordinates $x(u, v) = (\cos u\, \sin v, \sin u\, \sin v, \cos v)$, $0 \le u \le 2\pi, 0 \le v \le \pi$, Γ is described parametrically by

$$r(u, v) = \frac{a}{\cos u\, \sin v}\, x(u, v) = a \left(1, \tan u, \frac{1}{\cos u\, \tan v}\right). \tag{4.2}$$

Since the normal to the plane Γ is $\nu = (1, 0, 0)$, then (4.1) implies that $\sin u\, \sin v - \kappa\, m_2 = V_2(r(u, v))$ and $\cos v - \kappa\, m_3 = V_3(r(u, v))$. Hence V_2 and V_3 are univocally determined. Also, from (4.1) we get

$$V_1(r(u, v)) = \cos u\, \sin v - \kappa\, m_1 - \lambda(u, v). \tag{4.3}$$

Notice also that from (3.8),

$$\lambda = \nu \cdot (x - V) - \sqrt{(\nu \cdot (x - V))^2 - |x - V|^2 + \kappa^2},$$

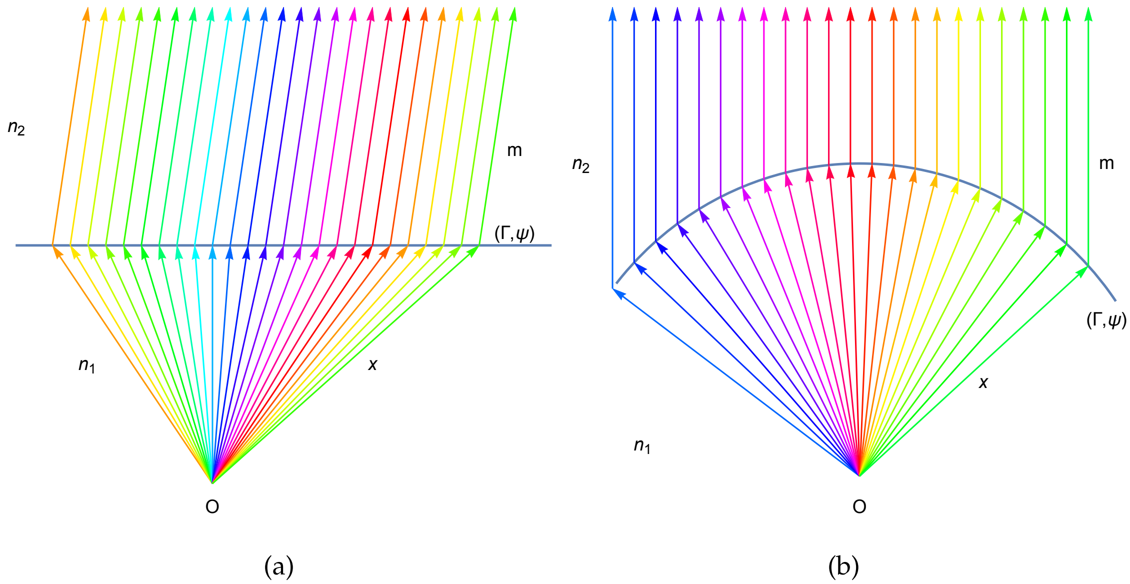

FIGURE 2. Planar and spherical metalenses

which in the present case yields

$$\begin{aligned}\lambda &= \cos u\ \sin v - V_1 - \sqrt{\kappa^2 - (\sin u\ \sin v - V_2)^2 - (\cos v - V_3)^2}\\ &= \cos u\ \sin v - V_1 - \sqrt{\kappa^2 - (\kappa\, m_2)^2 - (\kappa\, m_3)^2}\\ &= \cos u\ \sin v - V_1 - \kappa\, m_1 \qquad \text{since } m_1 > 0\\ &= \frac{a}{\sqrt{a^2 + x_2^2 + x_3^2}} - V_1(a, x_2, x_3) - \kappa\, m_1.\end{aligned}$$

This means that in (4.3) each V_1 determines λ and vice-versa.

We now write the field V in rectangular coordinates x_1, x_2, x_3. Since $\sqrt{a^2 + x_2^2 + x_3^2} = \dfrac{a}{\cos u\ \sin v}$, we can write

$$\begin{aligned}V_2(a, x_2, x_3) &= \frac{x_2}{\sqrt{a^2 + x_2^2 + x_3^2}} - \kappa\, m_2 = \frac{\partial}{\partial x_2}\sqrt{x_1^2 + x_2^2 + x_3^2}\Big|_{x_1=a} - \kappa\, m_2,\\ V_3(a, x_2, x_3) &= \frac{x_3}{\sqrt{a^2 + x_2^2 + x_3^2}} - \kappa\, m_3 = \frac{\partial}{\partial x_3}\sqrt{x_1^2 + x_2^2 + x_3^2}\Big|_{x_1=a} - \kappa\, m_3\\ V_1(a, x_2, x_3) &= \frac{a}{\sqrt{a^2 + x_2^2 + x_3^2}} - \kappa\, m_1 - \lambda = \frac{\partial}{\partial x_1}\sqrt{x_1^2 + x_2^2 + x_3^2}\Big|_{x_1=a} - \kappa\, m_1 - \lambda,\end{aligned}$$

for $-\infty < x_2, x_3 < \infty$. From (4.2) $u = \arctan(x_2/a)$ and $v = \arctan\left(\dfrac{\sqrt{a^2+x_2^2}}{x_3}\right)$, so $\lambda(u,v) = h(x_2,x_3)$. Let $\psi(x_1,x_2,x_3) = \sqrt{x_1^2+x_2^2+x_3^2} - \kappa\, m_1\, x_1 - \kappa\, m_2\, x_2 - \kappa\, m_3\, x_3$.

Therefore, if on the plane $x = a$ we give the field

$$V(x_1,x_2,x_3) := \nabla\psi(x_1,x_2,x_3) - h(x_2,x_3)\,\mathbf{i}, \tag{4.4}$$

then resulting metasurface does the desired refraction job. If we want V to be the gradient of a function, then $h(x_2,x_3)\,\mathbf{i}$ must be a gradient, which is only possible when $h(x_2,x_3) = C_0$ a constant; that is, $V = \nabla\left(\psi(x_1,x_2,x_3) - C_0\, x_1\right)$. As a particular case when $m_1 = 1$, $m_2 = m_3 = 0$, and $C_0 = 0$, we obtain the equivalent [YC14, Formula (2)] (where a different orientation of the coordinates is used) with $x_1 = a = f$. Notice also that if we want V in (4.4) to be tangential to the plane $x_1 = a$, that is, $(\nabla\psi(a,x_2,x_3) - h(x_2,x_3)\,\mathbf{i})\cdot(1,0,0) = 0$, then $h = \dfrac{a}{\sqrt{a^2+x_2^2+x_3^2}} - \kappa\, m_1$.

4.2. **Case of the sphere.** Now, the surface Γ considered is a sphere of radius R centered at the origin, that is, $r(u,v) = R\,x(u,v)$, with $x(u,v)$ spherical coordinates. We denote by $x = x(u,v)$; Figure 2(b). Since Γ is a sphere, the normal $\nu = x$ and from (4.1) we get $(x - \kappa\, m - V)\times x = 0$, so

$$(V + \kappa\, m)\times x = 0. \tag{4.5}$$

That is,

$$\begin{bmatrix} x_2 & -x_1 & 0 \\ -x_3 & 0 & x_1 \\ 0 & x_3 & -x_2 \end{bmatrix}\begin{pmatrix} V_1 + \kappa\, m_1 \\ V_2 + \kappa\, m_2 \\ V_3 + \kappa\, m_3 \end{pmatrix} = 0.$$

Notice that $\det\begin{bmatrix} x_2 & -x_1 & 0 \\ -x_3 & 0 & x_1 \\ 0 & x_3 & -x_2 \end{bmatrix} = 0$. Set $W_i = V_i + \kappa\, m_i$, so the system is equivalent to

$$\begin{bmatrix} 0 & 0 & 0 \\ x_2x_3 & -x_1x_3 & 0 \\ 0 & x_1x_3 & -x_1x_2 \end{bmatrix}\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = 0.$$

If $x_1x_2x_3 \neq 0$, the last matrix has rank two, so the space of solutions has dimension one and the solutions are given by

$$(W_1, W_2, W_3) = \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right) W_3,$$

with W_3 arbitrary. Therefore,

$$V_1\left(R\, x(u,v)\right) = \frac{x_1}{x_3}\left(V_3\left(R\, x(u,v)\right) + \kappa\, m_3\right) - \kappa\, m_1$$

$$V_2\left(R\, x(u,v)\right) = \frac{x_2}{x_3}\left(V_3\left(R\, x(u,v)\right) + \kappa\, m_3\right) - \kappa\, m_2,$$

with V_3 arbitrary.

Notice that if in (4.5) we take cross product with x, we get

$$\begin{aligned} 0 &= x \times ((V + \kappa\, m) \times x) \\ &= (V + \kappa\, m)\ (x \cdot x) - x\ ((V + \kappa\, m) \cdot x) \\ &= V + \kappa\, m - (\kappa\, (m \cdot x) + V \cdot x)\ x. \end{aligned}$$

Hence, if we want to pick V tangential to the sphere, we obtain

$$V(R\, x) = -\kappa\, m + \kappa\, (m \cdot x)\, x \text{ with } |x|= 1.$$

V is a field defined on the sphere of radius R. We shall determine a function ψ defined in a neighborhood of the sphere of radius R such that $V(Rx) = \nabla\psi(Rx)\big|_{|x|=1}$, and satisfying

$$\psi_{x_j}(R\, x) = -\kappa\, m_j + \kappa\, (m \cdot x)\, x_j, \text{ for } |x|= 1, \quad 1 \leq j \leq 3. \tag{4.6}$$

In fact, we have ($x = x(u, v)$)

$$\begin{aligned} \frac{\partial\psi(Rx(u, v))}{\partial u} &= R \sum_{k=1}^{3} \frac{\partial\psi}{\partial x_k}(Rx(u, v))\, (x_k)_u = R\, (D\psi)(Rx(u, v)) \cdot x_u \\ &= R\ (-\kappa\, m \cdot x_u + \kappa\, (m \cdot x)\, (x \cdot x_u)) = -\kappa\, R\, (m \cdot x_u) = -\kappa\, R\, \frac{\partial}{\partial u}(m \cdot x), \end{aligned}$$

and similarly,

$$\frac{\partial\psi(Rx(u, v))}{\partial v} = -\kappa\, R\, \frac{\partial}{\partial v}(m \cdot x).$$

Integrating the derivative in u yields

$$\psi(Rx(u, v)) = -\kappa\, R\, (m \cdot x) + g(v),$$

and integrating the derivate in v we obtain

$$\psi(Rx(u, v)) = -\kappa\, R\, (m \cdot x(u, v)) + C_1,$$

with C_1 an arbitrary constant. Writing this in rectangular coordinates yields

$$\psi\, (R\, (z_1, z_2, z_3)) = -\kappa\, R\, (m \cdot (z_1, z_2, z_3)) + C_1, \text{ for } |(z_1, z_2, z_3)|= 1.$$

We now define ψ on a neighborhood of $|z|= R$ so that (4.6) holds. Define

$$\psi(z) = -\kappa\, R\, (m \cdot z)\, |z|^{-1} + C_1, \text{ for } R - \epsilon < |z| < R + \epsilon. \tag{4.7}$$

We have

$$\nabla\psi(z) = -\kappa\, R\, m\, |z|^{-1} + \kappa\, R\, (m \cdot z)\, z\, |z|^{-3},$$

so for $z = Rx$, with $|x|= 1$, we obtain

$$\nabla\psi(Rx) = -\kappa\, m + \kappa\, (m \cdot x)\, x$$

as desired. Therefore the phase discontinuity ψ from (4.7) has gradient tangential to the sphere and can be placed on the spherical interface $|z|= R$ so that all rays from the origin are refracted into the fixed direction m.

5. Metalenses refracting into a set of variable directions

Suppose $m(u,v) = (m_1(u,v), m_2(u,v), m_3(u,v))$ is a given C^2 unit field of directions, and let Γ be a C^2 surface given parametrically by $r(u,v) = \rho(u,v)\, x(u,v)$ where $x(u,v)$ are spherical coordinates and $\rho(u,v) > 0$ is the polar radius. We want to see when is it possible to have a phase discontinuity ψ on the surface Γ so that each ray from the origin with direction $x(u,v)$ is refracted into the direction $m(u,v)$. From (4.1)

$$x(u,v) - \kappa\, m(u,v) - V(r(u,v)) = \lambda\, \nu(r(u,v))$$

so

$$(x - \kappa\, m - V) \times \nu = 0.$$

Taking cross product with ν yields

$$0 = \nu \times ((x - \kappa\, m - V) \times \nu) = (x - \kappa\, m - V)\,(\nu \cdot \nu) - \nu\, ((x - \kappa\, m - V) \cdot \nu)\,.$$

If V is tangential to Γ, then $V \cdot \nu = 0$ and so

$$0 = x - \kappa\, m - V - ((x - \kappa\, m) \cdot \nu)\, \nu,$$

that is,

$$V = x - \kappa\, m - ((x - \kappa\, m) \cdot \nu)\, \nu.$$

If $V(r(u,v)) = (\nabla\psi)(r(u,v))$, then

$$\psi_{x_j}(r(u,v)) = x_j(u,v) - \kappa\, m_j(u,v) - ((x(u,v) - \kappa\, m(u,v)) \cdot \nu(r(u,v)))\, \nu_j(r(u,v)).$$

Since $\nu \cdot r_u = \nu \cdot r_v = 0$ and $x \cdot x_u = x \cdot x_v = 0$,

$$\begin{aligned}
&\frac{\partial}{\partial u}\,(\psi(r(u,v)))\\
&= (\nabla\psi)(r(u,v)) \cdot r_u = (x - \kappa\, m) \cdot r_u - ((x - \kappa\, m) \cdot \nu)\ (\nu \cdot r_u)\\
&= (x - \kappa\, m) \cdot r_u = (x - \kappa\, m) \cdot (\rho_u\, x + \rho\, x_u)\\
&= \rho_u\, (x - \kappa\, m) \cdot x + \rho\, (x - \kappa\, m) \cdot x_u\\
&= \rho_u\, (1 - \kappa\, m \cdot x) - \kappa\, \rho\, m \cdot x_u = \rho_u\, (1 - \kappa\, m \cdot x) - \kappa\, \rho\, (m \cdot x)_u + \kappa\, \rho\, (m_u \cdot x)\\
&= \{\rho\, (1 - \kappa\, m \cdot x)\}_u + \kappa\, \rho\, (m_u \cdot x),
\end{aligned}$$

and similarly

$$\frac{\partial}{\partial v}\,(\psi(r(u,v))) = \{\rho\, (1 - \kappa\, m \cdot x)\}_v + \kappa\, \rho\, (m_v \cdot x).$$

Let us now consider the first order system in Φ

$$\begin{cases} \Phi_u & = \kappa\, \rho\, (m_u \cdot x) \\ \Phi_v & = \kappa\, \rho\, (m_v \cdot x). \end{cases} \tag{5.1}$$

where $\Phi(u,v) = \psi(r(u,v)) - \rho\ (1 - \kappa m \cdot x)$. If the given set of directions $m(u,v)$ and the surface Γ satisfy

$$m_u \cdot r_v = m_v \cdot r_u, \tag{5.2}$$

then by [Har02, Chapter 6, pp. 117-118](see also (6.17) below) there exists Φ solving (5.1). By integration we then obtain that the phase discontinuity ψ satisfies along Γ that

$$\psi(r(u,v)) = \rho\,(1-\kappa\, m\cdot x) + \Phi(u,v) = |r(u,v)| - \kappa\,(m(u,v)\cdot r(u,v)) + \Phi(u,v). \tag{5.3}$$

To find the gradient of ψ we need to have ψ defined in a neighborhood of the surface $r(u,v)$ such that (5.3) holds and that its gradient satisfies on $r(u,v)$

$$(\nabla\psi)(r(u,v)) = x-\kappa\, m - ((x-\kappa\, m)\cdot\nu)\;\nu. \tag{5.4}$$

Notice that this implies $(\nabla\psi)(r(u,v)) \perp \nu$. To construct the function ψ in a neighborhood of the surface Γ (we will construct it in a neighborhood of each point in Γ), given parametrically by $r(u,v)$, we use the notion of envelope from classical differential geometry; see for example [Pog59, Chapter 5, Section 4] or [dC76, Chapter 3]. We will actually construct a surface that is developable, in particular, it has Gauss curvature zero. For a recent reference on developable surfaces, its applications and design see [TBWP16].

Since the required ψ must satisfy (5.3), consider the surface Γ' given parametrically by

$$P(u,v) = \left(r(u,v), |r(u,v)| - \kappa\,(m(u,v)\cdot r(u,v)) + \Phi(u,v)\right) \tag{5.5}$$

in four dimensions. At each point $P(u,v)$, consider the 4-dimensional vector

$$N(u,v) = \left(x-\kappa\, m - ((x-\kappa\, m)\cdot\nu)\;\nu, -1\right),$$

where $x = x(u,v)$ and ν is the unit normal to the surface Γ at $r(u,v)$. Next consider the plane Π_{uv} passing through the point $P(u,v)$ and with normal $N(u,v)$, that is, in coordinates x_1, x_2, x_3, x_4, Π_{uv} has equation

$$F(x_1,x_2,x_3,x_4,u,v) := N(u,v)\cdot((x_1,x_2,x_3,x_4) - P(u,v)) = 0. \tag{5.6}$$

Therefore we have a family of planes Π_{uv} depending on the parameters u, v, and we will let $x_4 = \psi(x_1,x_2,x_3)$ be by definition the envelope to this family of planes. Of course, we need to know under what conditions on $r(u,v)$ and $m(u,v)$ this envelope ψ exists. It will be defined by solving the system of equations

$$\begin{cases} F(x_1,x_2,x_3,x_4,u,v) &= 0\\ \dfrac{\partial F}{\partial u}(x_1,x_2,x_3,x_4,u,v) &= 0\\ \dfrac{\partial F}{\partial v}(x_1,x_2,x_3,x_4,u,v) &= 0. \end{cases} \tag{5.7}$$

In fact, let us fix values $u = u_0, v = v_0$, and let $P_0 = P(u_0,v_0) = (p_1,p_2,p_3,p_4)$ be the corresponding value on the surface Γ'; and consider the map

$$G\,(x_1,x_2,x_3,x_4,u,v) = \left(F(x_1,x_2,x_3,x_4,u,v), \frac{\partial F}{\partial u}(x_1,x_2,x_3,x_4,u,v), \frac{\partial F}{\partial v}(x_1,x_2,x_3,x_4,u,v)\right).$$

The function G has continuous partial derivatives in a neighborhood of the point $(p_1, p_2, p_3, p_4, u_0, v_0)$, and

$$G(p_1, p_2, p_3, p_4, u_0, v_0) = 0.$$

By the implicit function theorem, if the Jacobian determinant
(5.8)

$$\frac{\partial G}{\partial(x_4, u, v)}(p_1, p_2, p_3, p_4, u_0, v_0) = \det \begin{pmatrix} \dfrac{\partial F}{\partial x_4} & \dfrac{\partial F}{\partial u} & \dfrac{\partial F}{\partial v} \\ \dfrac{\partial^2 F}{\partial x_4 \partial u} & \dfrac{\partial^2 F}{\partial u \partial u} & \dfrac{\partial^2 F}{\partial v \partial u} \\ \dfrac{\partial^2 F}{\partial x_4 \partial v} & \dfrac{\partial^2 F}{\partial u \partial v} & \dfrac{\partial^2 F}{\partial v \partial v} \end{pmatrix}\Bigg|_{(p_1,p_2,p_3,p_4,u_0,v_0)} \neq 0,$$

then there are unique differentiable functions g_1, g_2, g_3 in the variables x_1, x_2, x_3 defined in a neighborhood U of (p_1, p_2, p_3) such that $p_4 = g_1(p_1, p_2, p_3)$, $u_0 = g_2(p_1, p_2, p_3)$ and $v_0 = g_3(p_1, p_2, p_3)$ with

$$G(x_1, x_2, x_3, g_1(x_1, x_2, x_3), g_2(x_1, x_2, x_3), g_3(x_1, x_2, x_3)) = 0$$

for all $(x_1, x_2, x_3) \in U$. Therefore, if we let $\psi(x_1, x_2, x_3) = g_1(x_1, x_2, x_3)$ for $(x_1, x_2, x_3) \in U$, then ψ is the function we need, i.e., ψ is by construction defined in a neighborhood of the point $(p_1, p_2, p_3) \in \Gamma$ and satisfies (5.3) and (5.4).

We now analyze under what conditions on the surface Γ and m, (5.8) holds. Notice first that since $\partial_{x_4} F = -1$, the matrix inside the determinant in (5.8) equals

$$\begin{pmatrix} 1 & \dfrac{\partial F}{\partial u} & \dfrac{\partial F}{\partial v} \\ 0 & \dfrac{\partial^2 F}{\partial u \partial u} & \dfrac{\partial^2 F}{\partial v \partial u} \\ 0 & \dfrac{\partial^2 F}{\partial u \partial v} & \dfrac{\partial^2 F}{\partial v \partial v} \end{pmatrix},$$

and therefore (5.8) means

$$\det \begin{pmatrix} \dfrac{\partial^2 F}{\partial u \partial u} & \dfrac{\partial^2 F}{\partial v \partial u} \\ \dfrac{\partial^2 F}{\partial u \partial v} & \dfrac{\partial^2 F}{\partial v \partial v} \end{pmatrix} \neq 0.$$

Let us find what this means in terms of the initial surface Γ and the field m. To simplify the notation let $X = (x_1, x_2, x_3, x_4)$, so we can write (5.6) as

$$F(X, u, v) = N(u, v) \cdot (X - P(u, v)).$$

By calculation

$$(5.9)\qquad \begin{cases} F_u &= N_u \cdot (X-P) - N \cdot P_u \\ F_{uu} &= N_{uu} \cdot (X-P) - 2\,N_u \cdot P_u - N \cdot P_{uu} \\ F_{uv} &= N_{uv} \cdot (X-P) - N_u \cdot P_v - N_v \cdot P_u - N \cdot P_{uv} \\ F_{vv} &= N_{vv} \cdot (X-P) - 2\,N_v \cdot P_v - N \cdot P_{vv}. \end{cases}$$

We first show that

$$(5.10)\qquad N \cdot P_u = N \cdot P_v = 0.$$

Indeed, we have

$$P(u,v) = \rho(u,v)\ (x, 1-\kappa\, m \cdot x) + (0,\Phi),$$

so

$$(5.11)\qquad \begin{aligned} P_u &= \rho_u\ (x, 1-\kappa\, m\cdot x) + \rho\ (x_u, -\kappa\, m \cdot x_u - \kappa\, m_u \cdot x) + (0,\Phi_u) \\ P_v &= \rho_v\ (x, 1-\kappa\, m\cdot x) + \rho\ (x_v, -\kappa\, m \cdot x_v - \kappa\, m_v \cdot x) + (0,\Phi_v). \end{aligned}$$

Hence

$$\begin{aligned} N \cdot P_u &= \{\rho_u\ (x, 1-\kappa\, m\cdot x) + \rho\ (x_u, -\kappa\, m \cdot x_u - \kappa\, m_u \cdot x) + (0,\Phi_u)\} \cdot \\ &\qquad (x - \kappa\, m - ((x-\kappa\, m)\cdot \nu)\ \nu, -1) \\ &= (\rho_u\, x + \rho\, x_u) \cdot (x - \kappa\, m - ((x-\kappa\, m)\cdot\nu)\,\nu) - \rho_u\ (1-\kappa\, m \cdot x) + \\ &\qquad \rho\,(\,\kappa\, m\cdot x_u + \kappa\, m_u \cdot x) - \Phi_u \\ &= \rho_u - \rho_u\,\kappa\, x\cdot m - \rho\,\kappa\, x_u \cdot m - \rho_u\ + \rho_u\,\kappa\, m\cdot x + \rho\,\kappa\, m \cdot x_u + \rho\,\kappa\, m_u \cdot x - \rho\,\kappa\, m_u \cdot x \\ &= 0, \end{aligned}$$

since $(\rho_u\, x + \rho\, x_u)\cdot \nu = r_u \cdot \nu = 0$ and $x_u \cdot x = 0$. The same calculation with P_v instead of P_u yields the second identity in (5.10).

Next, differentiating (5.10) with respect to u and v yields

$$N \cdot P_{uu} = -N_u \cdot P_u, \qquad N \cdot P_{uv} = -N_u \cdot P_v = -N_v \cdot P_u, \qquad N\cdot P_{vv} = -N_v \cdot P_v,$$

since $P_{uv} = P_{vu}$. Hence letting $X = P$ in (5.11) yields

$$F_{uu} = -N_u \cdot P_u, \qquad F_{uv} = -N_v \cdot P_u = -N_u \cdot P_v, \qquad F_{vv} = -N_v \cdot P_v.$$

Now let us calculate these dot products. First set

$$B = (x - \kappa\, m) \cdot \nu$$

and write

$$
\begin{aligned}
&N_u \cdot P_u\\
&= \{\rho_u\ (x, 1-\kappa\, m\cdot x) + \rho\ (x_u, -\kappa\, m\cdot x_u) + (0, \Phi_u)\}\cdot\\
&\qquad \{x_u - \kappa\, m_u - [(x-\kappa\, m)\cdot \nu]_u\ \nu - [(x-\kappa\, m)\cdot\nu]\ \nu_u, 0\}\\
&= (\rho_u\, x + \rho\, x_u)\cdot(x_u - \kappa\, m_u - B_u\,\nu - B\,\nu_u)\\
&= (\rho_u\, x + \rho\, x_u)\cdot x_u - \kappa\ (\rho_u\, x + \rho\, x_u)\cdot m_u - B_u\ (\rho_u\, x + \rho\, x_u)\cdot \nu - B\ (\rho_u\, x + \rho\, x_u)\cdot\nu_u\\
&= \left(\sin^2 v\right)\rho - \kappa\ (\rho_u\, x + \rho\, x_u)\cdot m_u - B\ (\rho_u\, x + \rho\, x_u)\cdot \nu_u = \left(\sin^2 v\right)\rho - \kappa\, r_u\cdot m_u - B\, r_u\cdot\nu_u,
\end{aligned}
$$

since $x\cdot x_u = 0, x_u\cdot x_u = \sin^2 v$, and $(\rho_u\, x + \rho\, x_u)\cdot \nu = r_u\cdot\nu = 0$. Also $x_v\cdot x_v = 1$ and $x_u\cdot x_v = 0$, so we obtain similarly

$$N_v\cdot P_v = \rho - \kappa\, r_v\cdot m_v - B\, r_v\cdot \nu_v, \qquad N_u\cdot P_v = -\kappa\, r_u\cdot m_v - B\, r_u\cdot\nu_v.$$

Next, differentiating $r_u\cdot\nu = r_v\cdot\nu = 0$ yields

$$r_u\cdot\nu_u = -r_{uu}\cdot\nu, \qquad r_u\cdot\nu_v = -r_{uv}\cdot\nu, \qquad r_v\cdot\nu_v = -r_{vv}\cdot\nu. \tag{5.12}$$

Therefore

$$\begin{pmatrix} F_{uu} & F_{uv}\\ F_{vu} & F_{vv}\end{pmatrix} = \begin{pmatrix} -\left(\sin^2 v\right)\rho + \kappa\, r_u\cdot m_u - B\, r_{uu}\cdot\nu & \kappa\, r_u\cdot m_v - B\, r_{uv}\cdot\nu\\ \kappa\, r_u\cdot m_v - B\, r_{uv}\cdot\nu & -\rho + \kappa\, r_v\cdot m_v - B\, r_{vv}\cdot\nu\end{pmatrix},$$

and so

$$\det\begin{pmatrix} F_{uu} & F_{uv}\\ F_{vu} & F_{vv}\end{pmatrix} = \det\left[-\rho\begin{pmatrix} x_u\cdot x_u & x_u\cdot x_v\\ x_v\cdot x_u & x_v\cdot x_v\end{pmatrix} + \kappa\begin{pmatrix} r_u\cdot m_u & r_u\cdot m_v\\ r_v\cdot m_u & r_v\cdot m_v\end{pmatrix} - B\begin{pmatrix} r_{uu}\cdot\nu & r_{uv}\cdot\nu\\ r_{vu}\cdot\nu & r_{vv}\cdot\nu\end{pmatrix}\right], \tag{5.13}$$

with $B = (x-\kappa\, m)\cdot\nu$. Notice that the first and third matrices in the last determinant are respectively the first fundamental form of the 2-sphere, and the second fundamental form of the surface Γ.

Therefore, we have proved the following: if a variable field m and a surface Γ satisfy the compatibility condition (5.2), *and the determinant* (5.13) *is not zero at a point (u_0, v_0), then there is a neighborhood U of the point $r(u_0, v_0)$ and a phase discontinuity function ψ defined in U for the surface Γ, with gradient $\nabla\psi$ tangential to Γ, so that it yields the desired refraction job, i.e., each ray emanating in the direction $x(u, v)$, for (u, v) in a neighborhood of (u_0, v_0), is refracted by the metasurface (Γ, ψ) into the direction $m(u, v)$.*

Remark 5.1 (Case when m is a constant vector)**.** If $m(u, v) = (m_1, m_2, m_3)$ is constant, then (5.2) is clearly satisfied by any Γ and in condition (5.13) the second matrix on the right hand side is zero.

Remark 5.2. To illustrate the determinant condition (5.13), let us consider the special case when Γ is a sphere centered at the origin, and m is a constant vector. We have $r(u, v) = R\, x(u, v)$, and $\nu = x(u, v)$. So $r_{uu} = R x_{uu}$ and similarly for r_{vv} and r_{uv}. Also $B = 1 - \kappa\, m\cdot x$, $x_{uu}\cdot x = -\sin^2 v, x_{uv}\cdot x = 0$, and $x_{vv}\cdot x = -1$. Hence

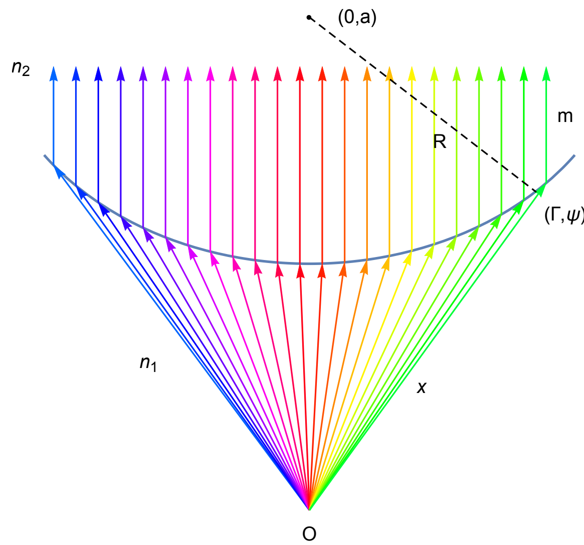

FIGURE 3. Off centered spherical metalens refracting into a fixed direction

$r_{uu} \cdot x = -R \sin^2 v$, $r_{uv} \cdot x = 0$, and $r_{vv} \cdot x = -R$. Therefore the determinant in (5.13) equals

$$R^2 \sin^2 v (1 - B)^2 = R^2 \kappa^2 \left(\sin^2 v\right) (m \cdot x)^2.$$

For example, if $m = (0, 0, 1)$, i.e., all rays are refracted vertically, then the determinant equals

$$R^2 \kappa^2 (\sin v \cos v)^2 = \frac{R^2 \kappa^2}{4} \sin^2(2v)$$

which is not zero as long as $v \neq \pi/2$ or zero. This shows also that for the sphere the phase discontinuity ψ exists and can be obtained by solving the system of equations (5.7). Notice that in this case a phase discontinuity ψ was calculated explicitly in Section 4.2 and given by (4.7).

Remark 5.3 (Case when Γ is off centered)**.** A case considered in [AKG+12, Section 3] is when a sphere of radius R is centered at a point $(0, 0, a)$ with $a > R$, and the authors claim there that it is not possible to find a phase discontinuity on such a sphere so that all rays from the origin are refracted into the vertical direction. We believe this claim is in error and in fact, with the method above will show that for each unit $m = (m_1, m_2, m_3)$ with $m_3 > 0$, there is a phase discontinuity ψ defined in a neighborhood of such a sphere so that its gradient is tangential to the sphere and so that radiation from the origin is refracted into a fixed direction m, see Figure 3. In particular, when m is vertical a phase discontinuity exists. By reversibility of optical paths, this shows that the conclusion in [AKG+12, Section 3] is incorrect.

First, the lower part of the sphere with center at $(0,0,a)$ and radius R is parametrized by the vector $r(u,v)=\rho(u,v)\,x(u,v)$ with

$$\rho(u,v)=a\,\cos v-\sqrt{R^2-a^2\,\sin^2 v},$$

where $0\le v\le \arcsin(R/a)$; and the unit normal to the sphere pointing upwards is

$$\nu=\frac{(0,0,a)-\rho(u,v)\,x(u,v)}{R}.$$

To show our claim, we need to verify that the determinant in (5.13) is not zero. From (5.12) we obtain by simple calculations that

$$\begin{aligned} r_{uu}\cdot\nu &= -r_u\cdot\nu_u=\frac{1}{R}\left(\sin^2 v\right)\,\rho^2\\ r_{uv}\cdot\nu &= -r_u\cdot\nu_v=\frac{1}{R}\,\rho_u\,\rho_v=0\\ r_{vv}\cdot\nu &= -r_v\cdot\nu_v=\frac{1}{R}\,\left((\rho_v)^2+\rho^2\right). \end{aligned}$$

Therefore the determinant in (5.13) equals

$$\det\begin{pmatrix}F_{uu} & F_{uv}\\ F_{vu} & F_{vv}\end{pmatrix}=\rho\,(\sin^2 v)\,\left(1+\frac{B}{R}\,\rho\right)\left(\rho+\frac{B}{R}\left(\rho^2+(\rho_v)^2\right)\right), \tag{5.14}$$

with

$$B=(x-\kappa\,m)\cdot\nu=\frac{1}{R}\,(x-\kappa\,m)\cdot((0,0,a)-\rho\,x)=\frac{1}{R}\left(\sqrt{R^2-a^2\,\sin^2 v}-\kappa\,a\,m_3+\kappa\,\rho\,(m\cdot x)\right).$$

The last determinant is not zero for u,v such that

$$\sin^2 v\neq 0,\quad 1+\frac{B}{R}\,\rho\neq 0,\quad \text{and } \rho+\frac{B}{R}\left(\rho^2+(\rho_v)^2\right)\neq 0.$$

Let us take for example $m=(0,0,1)$, i.e., rays are refracted vertically, then we get

$$B=\frac{1}{R}\left((1-\kappa\,\cos v)\,\sqrt{R^2-a^2\,\sin^2 v}-\kappa\,a\,\sin^2 v\right),$$

so B is independent of u. If $v\approx 0$, then $B\approx 1-\kappa$, $\rho\approx a-R$ and $\rho_v\approx 0$, so

$$\begin{aligned} 1+\frac{B}{R}\,\rho &\approx 1+(1-\kappa)\left(\frac{a}{R}-1\right)\\ \rho+\frac{B}{R}\left(\rho^2+(\rho_v)^2\right) &\approx (a-R)\left(1+(1-\kappa)\left(\frac{a}{R}-1\right)\right). \end{aligned}$$

Recall that $\kappa=n_2/n_1$. If $\kappa<1$, since $a>R$, we obtain that $1+(1-\kappa)\left(\dfrac{a}{R}-1\right)\neq 0$. If $\kappa>1$, then $1+(1-\kappa)\left(\dfrac{a}{R}-1\right)\neq 0$ if and only if $\kappa\neq 1+\dfrac{R}{a-R}$. This shows that in these cases the determinant in (5.14) is not zero for $v\neq 0$ with v close to zero. Therefore there exists a phase discontinuity ψ, on the sphere centered at $(0,0,a)$ with radius R, defined in a neighborhood of each point of the form $\rho(u,v)\,x(u,v)$ with v close to zero.

6. Given a phase discontinuity find an admissible surface

We now turn to the second question proposed at the beginning of Section 4, that is, of finding the surface Γ when the field $V = (V_1, V_2, V_3)$ is given. The unknown surface is given parametrically by

$$r(u,v) = \rho(u,v)\, x(u,v)$$

where $x(u,v)$ are spherical coordinates as before, and we seek the polar radius ρ; the value of V along the surface is $V(r(u,v))$. From (4.1), $x(u,v) - \kappa\, m - V(r(u,v))$ is a multiple of the normal ν at $r(u,v)$, so

$$r_u(u,v)\cdot(x(u,v) - \kappa\, m - V(r(u,v))) = 0 \quad \text{and} \quad r_v(u,v)\cdot(x(u,v) - \kappa\, m - V(r(u,v))) = 0.$$

We have

$$\begin{aligned} r_u(u,v) &= [\rho(u,v)]_u\, x(u,v) + \rho(u,v)\, x_u(u,v),\\ r_v(u,v) &= [\rho(u,v)]_v\, x(u,v) + \rho(u,v)\, x_v(u,v), \end{aligned}$$

so

$$\begin{aligned} 0 &= r_u(u,v)\cdot(x(u,v) - \kappa\, m - V(r(u,v)))\\ &= \Big([\rho(u,v)]_u\, x(u,v) + \rho(u,v)\, x_u(u,v)\Big)\cdot(x(u,v) - \kappa\, m - V(r(u,v)))\\ &= [\rho(u,v)]_u\,(1 - x(u,v)\cdot[\kappa\, m + V(r(u,v))]) - \rho(u,v)\, x_u(u,v)\cdot[\kappa\, m + V(r(u,v))]\,, \end{aligned}$$

and a similar equation for r_v. That is, $\rho(u,v)$ satisfies the first order nonlinear system of pdes (depending on V)‡:

$$\begin{cases} \rho_u(u,v) - \dfrac{x_u\cdot[\kappa\, m + V(\rho(u,v)\, x(u,v))]}{1 - x(u,v)\cdot[\kappa\, m + V(\rho(u,v)\, x(u,v))]}\,\rho(u,v) = 0\\[2ex] \rho_v(u,v) - \dfrac{x_v\cdot[\kappa\, m + V(\rho(u,v)\, x(u,v))]}{1 - x(u,v)\cdot[\kappa\, m + V(\rho(u,v)\, x(u,v))]}\,\rho(u,v) = 0. \end{cases} \tag{6.15}$$

If $F = (F_1, F_2)$ with

$$\begin{aligned} F_1(u,v,\rho) &= \frac{x_u\cdot[\kappa\, m + V(\rho\, x(u,v))]}{1 - x(u,v)\cdot[\kappa\, m + V(\rho\, x(u,v))]}\,\rho\\ F_2(u,v,\rho) &= \frac{x_v\cdot[\kappa\, m + V(\rho\, x(u,v))]}{1 - x(u,v)\cdot[\kappa\, m + V(\rho\, x(u,v))]}\,\rho, \end{aligned}$$

then (6.15) can be written as

$$\nabla\rho = F(u,v,\rho). \tag{6.16}$$

To solve the system (6.16) we need an initial condition, say $\rho(u_0,v_0) = \rho_0$, and use a result from [Har02, Chapter 6, pp. 117-118], that is, if

$$\frac{\partial F_1}{\partial v}(u,v,\rho) + \frac{\partial F_1}{\partial \rho}(u,v,\rho)F_2(u,v,\rho) = \frac{\partial F_2}{\partial u}(u,v,\rho) + \frac{\partial F_2}{\partial \rho}(u,v,\rho)F_1(u,v,\rho) \tag{6.17}$$

‡We are assuming that $1 - x(u,v)\cdot[\kappa\, m + V(\rho(u,v)\, x(u,v))] \neq 0$.

holds for all (u,v,ρ) in an open set O, then for each $(u_0,v_0,\rho_0) \in O$ there is neighborhood U of (u_0,v_0) and a unique solution $\rho(u,v)$ defined for $(u,v) \in U$ solving the system (6.16) and satisfying $\rho(u_0,v_0) = \rho_0$.

We will see under what circumstances on the field V condition (6.17) is satisfied, and therefore the existence of the desired surface $r(u,v)$ will be guaranteed. Set

$$W(u,v,\rho) = \kappa\, m + V\left(\rho\, x(u,v)\right), \tag{6.18}$$

then

$$F_1(u,v,\rho) = \frac{x_u \cdot W(u,v,\rho)}{1 - x(u,v)\cdot[W(u,v,\rho)]}\,\rho, \qquad F_2(u,v,\rho) = \frac{x_v \cdot W(u,v,\rho)}{1 - x(u,v)\cdot[W(u,v,\rho)]}\,\rho.$$

We have

$$\begin{aligned}
\frac{\partial F_1}{\partial v} &= \left\{(x_{uv}\cdot W + x_u \cdot W_v)\,(1 - x\cdot W)^{-1} + (x_v \cdot W + x\cdot W_v)\,(x_u\cdot W)\,(1-x\cdot W)^{-2}\right\}\rho\\
\frac{\partial F_2}{\partial u} &= \left\{(x_{vu}\cdot W + x_v \cdot W_u)\,(1 - x\cdot W)^{-1} + (x_u \cdot W + x\cdot W_u)\,(x_v\cdot W)\,(1-x\cdot W)^{-2}\right\}\rho\\
\frac{\partial F_1}{\partial \rho} &= (x_u\cdot W)\,(1-x\cdot W)^{-1} + \left\{\left(x_u\cdot W_\rho\right)(1-x\cdot W)^{-1} + (x_u\cdot W)\,(x\cdot W_\rho)\,(1-x\cdot W)^{-2}\right\}\rho\\
\frac{\partial F_2}{\partial \rho} &= (x_v\cdot W)\,(1-x\cdot W)^{-1} + \left\{\left(x_v\cdot W_\rho\right)(1-x\cdot W)^{-1} + (x_v\cdot W)\,(x\cdot W_\rho)\,(1-x\cdot W)^{-2}\right\}\rho.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial F_1}{\partial v} - \frac{\partial F_2}{\partial u} = \Big\{&(x_u\cdot W_v - x_v\cdot W_u)\,(1-x\cdot W)^{-1}\\
&+ \left((x\cdot W_v)\,(x_u\cdot W) - (x\cdot W_u)\,(x_v\cdot W)\right)\,(1-x\cdot W)^{-2}\Big\}\rho
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial F_1}{\partial \rho}F_2 - \frac{\partial F_2}{\partial \rho}F_1
&= \Big[(x_u\cdot W)\,(1-x\cdot W)^{-1}\\
&\qquad + \left\{\left(x_u\cdot W_\rho\right)(1-x\cdot W)^{-1} + (x_u\cdot W)\,(x\cdot W_\rho)\,(1-x\cdot W)^{-2}\right\}\rho\Big]\,(x_v \cdot W)\,(1-x\cdot W)^{-1}\\
&\quad - \Big[(x_v\cdot W)\,(1-x\cdot W)^{-1}\\
&\qquad + \left\{\left(x_v\cdot W_\rho\right)(1-x\cdot W)^{-1} + (x_v\cdot W)\,(x\cdot W_\rho)\,(1-x\cdot W)^{-2}\right\}\rho\Big]\,(x_u \cdot W)\,(1-x\cdot W)^{-1}\\
&= \Big((x_u\cdot W_\rho)\,(x_v\cdot W) - (x_v\cdot W_\rho)\,(x_u\cdot W)\Big)\,(1-x\cdot W)^{-2}\,\rho.
\end{aligned}$$

Therefore (6.17) holds if

$$\begin{aligned}
&\frac{\partial F_1}{\partial v} - \frac{\partial F_2}{\partial u} + \frac{\partial F_1}{\partial \rho}F_2 - \frac{\partial F_2}{\partial \rho}F_1 \\
&= \Big\{(x_u \cdot W_v - x_v \cdot W_u)\,(1 - x \cdot W)^{-1} + ((x \cdot W_v)\,(x_u \cdot W) - (x \cdot W_u)\,(x_v \cdot W))\;(1 - x \cdot W)^{-2}\Big\}\;\rho \\
&\qquad + \Big((x_u \cdot W_\rho)\,(x_v \cdot W) - (x_v \cdot W_\rho)\,(x_u \cdot W)\Big)\;(1 - x \cdot W)^{-2}\;\rho = 0.
\end{aligned}$$

Since we assume $1 - x \cdot W \neq 0$ and $\rho > 0$, this is equivalent to

$$\begin{aligned}
&(x_u \cdot W_v - x_v \cdot W_u)\,(1 - x \cdot W) + ((x \cdot W_v)\,(x_u \cdot W) - (x \cdot W_u)\,(x_v \cdot W)) \\
&\qquad + \Big((x_u \cdot W_\rho)\,(x_v \cdot W) - (x_v \cdot W_\rho)\,(x_u \cdot W)\Big) = 0,
\end{aligned}$$

that is,

$$\begin{aligned}
&(x_u \cdot W_v - x_v \cdot W_u)\,(1 - x \cdot W) \\
&\quad + \Big(\Big((x \cdot W_v) - (x_v \cdot W_\rho)\Big)\;(x_u \cdot W) - \Big((x \cdot W_u) - (x_u \cdot W_\rho)\Big)\;(x_v \cdot W)\Big) = 0.
\end{aligned} \tag{6.19}$$

We have

$$\begin{aligned}
W_u &= \rho\;(\nabla V_1 \cdot x_u, \nabla V_2 \cdot x_u, \nabla V_3 \cdot x_u) \\
W_v &= \rho\;(\nabla V_1 \cdot x_v, \nabla V_2 \cdot x_v, \nabla V_3 \cdot x_v) \\
W_\rho &= (\nabla V_1 \cdot x, \nabla V_2 \cdot x, \nabla V_3 \cdot x)\,.
\end{aligned}$$

Now

$$x \cdot W_v = \rho\,\sum_{k=1}^{3} x_k\,(\nabla V_k \cdot x_v) = \rho\,\sum_{k=1}^{3} x_k\,\sum_{j=1}^{3}\frac{\partial V_k}{\partial y_j}(x_j)_v = \rho\,\sum_{k,j=1}^{3}\frac{\partial V_k}{\partial y_j}(x_j)_v\,x_k.$$

If we let

$$A = \begin{pmatrix} \frac{\partial V_1}{\partial y_1} & \frac{\partial V_1}{\partial y_2} & \frac{\partial V_1}{\partial y_3} \\ \frac{\partial V_2}{\partial y_1} & \frac{\partial V_2}{\partial y_2} & \frac{\partial V_2}{\partial y_3} \\ \frac{\partial V_3}{\partial y_1} & \frac{\partial V_3}{\partial y_2} & \frac{\partial V_3}{\partial y_3} \end{pmatrix},$$

then

$$x \cdot W_v = \rho\,x\,A\,(x_v)^t$$

where x, x_v are row vectors and t denotes the transpose. Similarly

$$\begin{aligned}
x \cdot W_u &= \rho\,x\,A\,(x_u)^t \quad x_u \cdot W_v = \rho\,x_u\,A\,(x_v)^t \\
x_v \cdot W_u &= \rho\,x_v\,A\,(x_u)^t \quad x_u \cdot W_\rho = x_u\,A\,(x)^t \quad x_v \cdot W_\rho = x_v\,A\,(x)^t.
\end{aligned}$$

Suppose $V = \nabla\psi$, then $A = \nabla^2\psi$ is a symmetric matrix, so

$$x_u \cdot W_v = x_v \cdot W_u$$

$$(x \cdot W_v) - (x_v \cdot W_\rho) = (\rho - 1)\, x\, A\, (x_v)^t = \frac{\rho - 1}{\rho}\, (x \cdot W_v)$$

$$(x \cdot W_u) - (x_u \cdot W_\rho) = (\rho - 1)\, x\, A\, (x_u)^t = \frac{\rho - 1}{\rho}\, (x \cdot W_u)$$

and (6.19) reads

$$(\rho - 1)\, \left\{ (x\, A\, (x_v)^t)\, (x_u \cdot W) - (x\, A\, (x_u)^t)\, (x_v \cdot W) \right\} = 0,^{\S} \tag{6.20}$$

which can be written as

$$\det \begin{pmatrix} x\, A\, (x_u)^t & x\, A\, (x_v)^t \\ x_u \cdot W & x_v \cdot W \end{pmatrix} = \det \begin{pmatrix} x_u \cdot Ax & x_v \cdot Ax \\ x_u \cdot W & x_v \cdot W \end{pmatrix} = 0.$$

From the Cauchy-Binet formula for cross products[¶], this means that

$$(x_u \times x_v) \cdot (Ax \times W) = 0$$

and since $x_u \times x_v \parallel x$, (6.20) is equivalent to the following geometric condition:

$$x \cdot (Ax \times W) = 0.^{\|} \tag{6.21}$$

Therefore, if the field $V = \nabla\psi$, *W is given in* (6.18), *and* (6.20) *(or equivalently* (6.21)*) holds in an open set* O *in the variables* (ρ, u, v), *then for each* $(\rho_0, u_0, v_0) \in O$ *the system* (6.15) *has a unique solution* $\rho(u, v)$ *defined in a neighborhood of* (u_0, v_0) *and satisfying the initial condition* $\rho(u_0, v_0) = \rho_0$. Notice that if $V = V_0$ is a constant field, then $A = 0$ and so (6.20) obviously holds. In this case, (6.15) can be easily integrated and the solution is

$$\rho(u, v) = \frac{C_1}{1 - x(u, v) \cdot (\kappa\, m + V_0)} + C_2$$

with C_i constants.

Notice also that with the choice V as in (4.4), with $h \neq 0$ so $1 - x \cdot W \neq 0$, the system of equations (6.15) becomes

$$\begin{cases} \rho_u(u, v) - \dfrac{\sin u}{\cos u}\, \rho(u, v) = 0 \\[2ex] \rho_v(u, v) + \dfrac{\cos v}{\sin v}\, \rho(u, v) = 0, \end{cases}$$

whose solution is $\rho(u, v) = \dfrac{C}{\cos u\, \sin v}$, where the constant C is determined by the point where the solution passes through. This is in agreement with (4.2).

[§] Since $(x \cdot W)_u = x_u \cdot W + \rho\, x\, A\, (x_u)^t$ and similarly for $(x \cdot W)_v$, this condition can be re-written as $(\rho - 1)\, \{x\, A\, (x_v)^t\, (x \cdot W)_u - x\, A\, (x_u)^t\, (x \cdot W)_v\} = 0$.

[¶] $(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$.

[‖] Equivalently $W \cdot (Ax \times x) = Ax \cdot (x \times W) = 0$.

7. Near field refracting metasurfaces

The near field case can be regarded as a special case from Section 5 when the vector field $m(u,v)$ points towards a fixed point Q, and therefore the method from that section can be used to derive conditions for the existence of the desired metasurface. In fact, if the surface Γ is parametrized by $r(u,v)$ and $m(u,v) = \dfrac{Q - r(u,v)}{|Q - r(u,v)|}$, then it is easy to see that the compatibility condition (5.2) holds. The existence of the phase discontinuity then follows when the determinant in (5.13) is not zero.

However, the phase discontinuities in the planar and spherical cases can be obtained explicitly as follows; see Figure 4.

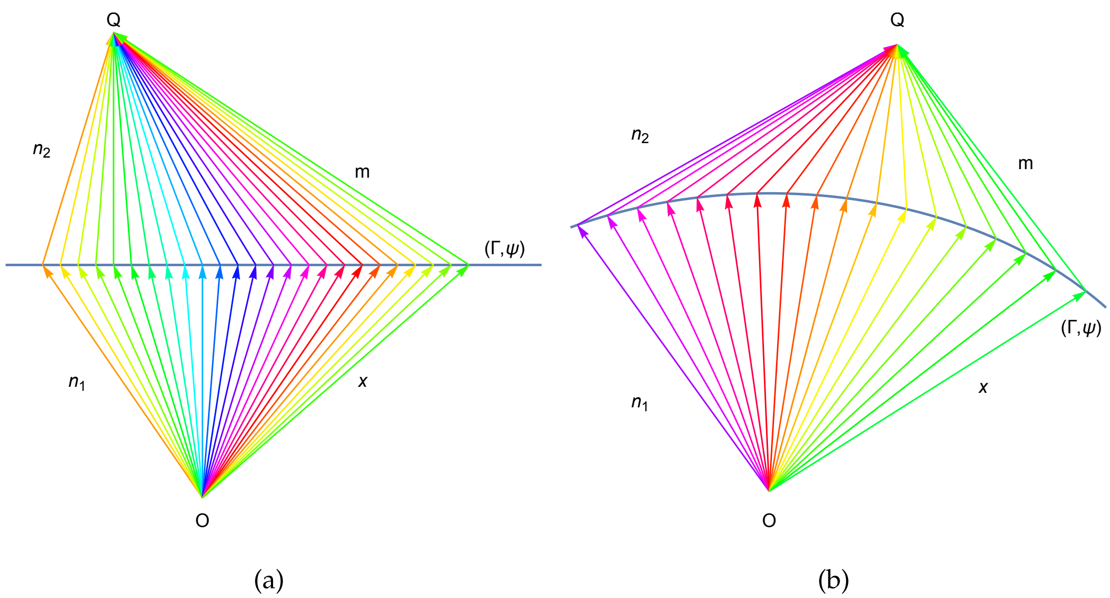

Figure 4. Planar and spherical metalenses in the near field

7.1. **Case of a plane interface.** Let O be the origin in medium I with index n_1 and let $Q = (q_1, q_2, q_3)$ be a point in medium II with index n_2. Denote by Γ the plane with equation $x_1 = a$ so that it separates the points O and Q. We find the field V so that rays from O are refracted into Q. We know from Section 4.1 that Γ is given parametrically by (4.2); the normal $\nu = (1, 0, 0)$. So we seek V such that (4.1) holds. Since the refracted vector from each point $r(u,v)$ on the plane interface to the point Q has unit direction $\dfrac{Q - r(u,v)}{|Q - r(u,v)|}$, V must satisfy

$$\begin{aligned}
\cos u \, \sin v - \kappa \, \frac{q_1 - a}{|Q - r(u,v)|} &= \lambda + V_1 \\
\sin u \, \sin v - \kappa \, \frac{q_2 - a \, \tan u}{|Q - r(u,v)|} &= V_2 \\
\cos v - \kappa \, \frac{q_3 - a/\cos u \tan v}{|Q - r(u,v)|} &= V_3.
\end{aligned}$$

Re writing these equations in rectangular coordinates yields

$$\begin{aligned}
\frac{a}{\sqrt{a^2 + x_2^2 + x_3^2}} - \kappa \, \left.\frac{q_1 - a}{|Q - (x_1, x_2, x_3)|}\right|_{x_1 = a} &= \lambda + V_1 \\
\frac{x_2}{\sqrt{a^2 + x_2^2 + x_3^2}} - \kappa \, \left.\frac{q_2 - x_2}{|Q - (x_1, x_2, x_3)|}\right|_{x_1 = a} &= V_2 \\
\frac{x_3}{\sqrt{a^2 + x_2^2 + x_3^2}} - \kappa \, \left.\frac{q_3 - x_3}{|Q - (x_1, x_2, x_3)|}\right|_{x_1 = a} &= V_3.
\end{aligned}$$

Therefore, V_i, $i = 1, 2, 3$, are determined:

$$\begin{aligned}
V_1(a, x_2, x_3) &= \partial_{x_1}\left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right)\Big|_{x_1 = a} + \kappa \, \frac{\partial}{\partial x_1}|Q - (x_1, x_2, x_3)|\Big|_{x_1 = a} - \lambda \\
V_2(a, x_2, x_3) &= \partial_{x_2}\left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right)\Big|_{x_1 = a} + \kappa \, \frac{\partial}{\partial x_2}|Q - (x_1, x_2, x_3)|\Big|_{x_1 = a} \\
V_3(a, x_2, x_3) &= \partial_{x_3}\left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right)\Big|_{x_1 = a} + \kappa \, \frac{\partial}{\partial x_3}|Q - (x_1, x_2, x_3)|\Big|_{x_1 = a},
\end{aligned}$$

where λ is chosen arbitrarily. Notice that if we let

$$\psi(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2} + \kappa \, |Q - (x_1, x_2, x_3)|$$

and choose $\lambda = 0$, then $V = \nabla\psi$, and so the plane with the phase discontinuity function ψ does the desired refraction job.

7.2. **Case of a spherical interface.** If Γ is the sphere of radius R centered at the origin, that is, $r(u,v) = R\, x(u,v)$, then the normal $\nu = x$, and from (4.1) we get

$$\left(x - \kappa \, \frac{Q - r(u,v)}{|Q - r(u,v)|} - V\right) \times x = 0.$$

As before taking cross product with x yields

$$V + \kappa \, \frac{Q - r(u,v)}{|Q - r(u,v)|} - \left(\kappa \left(\frac{Q - r(u,v)}{|Q - r(u,v)|} \cdot x\right) + V \cdot x\right) x = 0.$$

Assuming V is tangential to the sphere,

$$V = -\kappa \frac{Q - r(u,v)}{|Q - r(u,v)|} + \kappa \left(\frac{Q - r(u,v)}{|Q - r(u,v)|} \cdot x \right) x.$$

If $V(Rx(u,v)) = (\nabla\psi)(Rx(u,v))$, then

$$\psi_{x_j}(Rx(u,v)) = -\kappa \frac{q_j - Rx_j(u,v)}{|Q - Rx(u,v)|} + \kappa \left(\frac{Q - Rx(u,v)}{|Q - Rx(u,v)|} \cdot x \right) x_j, \qquad j = 1,2,3. \tag{7.1}$$

Hence

$$\frac{\partial}{\partial u} (\psi(Rx(u,v))) = (\nabla\psi)(Rx(u,v)) \cdot Rx_u = -\kappa R \frac{Q - Rx(u,v)}{|Q - Rx(u,v)|} \cdot x_u, \tag{7.2}$$

and similarly

$$\frac{\partial}{\partial v} (\psi(Rx(u,v))) = -\kappa R \frac{Q - Rx(u,v)}{|Q - Rx(u,v)|} \cdot x_v, \tag{7.3}$$

since $x \cdot x_u = x \cdot x_v = 0$. Since ψ is assumed C^2, we get

$$\left(\frac{Q - Rx(u,v)}{|Q - Rx(u,v)|} \right)_u \cdot x_v = \left(\frac{Q - Rx(u,v)}{|Q - Rx(u,v)|} \right)_v \cdot x_u. \tag{7.4}$$

Integrating (7.2) in u yields

$$\psi(Rx(u,v)) = -\kappa R \int \frac{Q - Rx(u',v)}{|Q - Rx(u',v)|} \cdot x_u(u',v)\, du' + h(v),$$

for some function h. To calculate h, we differentiate the integral with respect to v and use (7.4):

$$\begin{aligned}
&\frac{\partial}{\partial v} (\psi(Rx(u,v))) \\
&= -\kappa R \int \frac{\partial}{\partial v} \left(\frac{Q - Rx(u',v)}{|Q - Rx(u',v)|} \cdot x_u(u',v) \right) du' + h'(v) \\
&= -\kappa R \int \left\{ \frac{\partial}{\partial v} \left(\frac{Q - Rx(u',v)}{|Q - Rx(u',v)|} \right) \cdot x_u(u',v) + \frac{Q - Rx(u',v)}{|Q - Rx(u',v)|} \cdot x_{uv}(u',v) \right\} du' + h'(v) \\
&= -\kappa R \int \left\{ \frac{\partial}{\partial u} \left(\frac{Q - Rx(u',v)}{|Q - Rx(u',v)|} \right) \cdot x_v(u',v) + \frac{Q - Rx(u',v)}{|Q - Rx(u',v)|} \cdot x_{vu}(u',v) \right\} du' + h'(v) \\
&= -\kappa R \int \frac{\partial}{\partial u} \left(\frac{Q - Rx(u',v)}{|Q - Rx(u',v)|} \cdot x_v(u',v) \right) du' + h'(v) \\
&= -\kappa R \left(\frac{Q - Rx(u,v)}{|Q - Rx(u,v)|} \cdot x_v(u,v) \right) + h'(v)
\end{aligned}$$

which implies $h'(v) = 0$ from (7.3). Therefore, the phase discontinuity ψ on the sphere satisfies

$$\begin{aligned}\psi(Rx(u,v)) &= -\kappa R \int \frac{Q - Rx(u',v)}{|Q - Rx(u',v)|} \cdot x_u(u',v)\, du' + C \\ &= \kappa \int \partial_u(|Q - Rx(u',v)|)\, du' + C = \kappa\, |Q - Rx(u,v)| + C\end{aligned}$$

with C a constant. Writing this in rectangular coordinates yields

$$\psi(R(z_1,z_2,z_3)) = \kappa\, |Q - R(z_1,z_2,z_3)| + C, \qquad \text{for } |(z_1,z_2,z_3)| = 1.$$

We now define ψ on a neighborhood of $|z| = R$ so that (7.1) holds. Let

$$\psi(z) = \kappa \left| Q - R\frac{z}{|z|} \right| + C, \ \text{ for } R - \epsilon < |z| < R + \epsilon. \tag{7.5}$$

We have

$$\nabla\psi(z) = -\kappa R \frac{Q - R\frac{z}{|z|}}{\left|Q - R\frac{z}{|z|}\right|} \frac{1}{|z|} + \kappa R \left(\frac{Q - R\frac{z}{|z|}}{\left|Q - R\frac{z}{|z|}\right|} \cdot \frac{z}{|z|} \right) \frac{z}{|z|^2},$$

so for $z = Rx$, with $|x| = 1$, we obtain

$$\nabla\psi(Rx) = -\kappa \frac{Q - Rx}{|Q - Rx|} + \kappa \left(\frac{Q - Rx}{|Q - Rx|} \cdot x \right) x$$

as desired. Therefore the phase discontinuity ψ in (7.5) has gradient tangential to the sphere and can be placed on the spherical interface $|z| = R$ so that all rays from the origin are refracted into the point Q.

8. Conclusion

A rigorous mathematical foundation of general metasurfaces is provided. The starting point is the derivation of a generalized Snell's law in the presence of a phase discontinuity using wavefronts. This is used also to derive all possible critical angles. We solve, under appropriate curvature type conditions on the surface Γ, the problem of finding a phase discontinuity, so that the pair (surface and phase discontinuity) refracts light in a desired manner. When a phase discontinuity is given, we derive conditions so that a surface is admissible for that phase discontinuity in the far field setting. Extensions to the case when the far field is a set of variable directions are given, and examples and explicit calculations of phase discontinuities are also provided. The near field case is also studied.

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