Horava-Lifshitz theory as a fermionic aether in Ashtekar gravity

Stephon Alexander
Haverford College

Joao Magueijo

Antonino Marciano

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Stephon Alexander[*]

Department of Physics and Astronomy, Dartmouth College, Hanover, New Hampshire 03755, USA

João Magueijo[†]

Theoretical Physics, Blackett Laboratory, Imperial College, London, SW7 2BZ, United Kingdom

Antonino Marcianò[‡]

Department of Physics, The Koshland Integrated Natural Science Center, Haverford College, Haverford, Pennsylvania 19041, USA
and Department of Physics, Princeton University, New Jersey 08544, USA

We show how Hořava-Lifshitz (HL) theory appears naturally in the Ashtekar formulation of relativity if one postulates the existence of a fermionic field playing the role of aether. The spatial currents associated with this field must be switched off for the equivalence to work. Therefore the field supplies the preferred frame associated with breaking refoliation (time diffeomorphism) invariance, but obviously the symmetry is only spontaneously broken if the field is dynamic. When Dirac fermions couple to the gravitational field via the Ashtekar variables, the low energy limit of HL gravity, recast in the language of Ashtekar variables, naturally emerges (provided the spatial fermion current identically vanishes). HL gravity can therefore be interpreted as a timelike current, or a Fermi aether, that fills space-time, with the Immirzi parameter, a chiral fermionic coupling, and the fermionic charge density fixing the value of the parameter λ determining HL theory. This reinterpretation sheds light on some features of HL theory, namely its good convergence properties.

I. INTRODUCTION

While there are stringent experimental constraints on breaking local Lorentz invariance in particle physics, it is well known that diffeomorphism invariance plays a more prominent structural role in general relativity and quantum gravity since it is possible that near the Planck scale, Lorentz symmetry is not fundamental. One of our best tests of Lorentz invariance on large-distance scales is the cosmic microwave background, which breaks Lorentz invariance by choosing a preferred timelike frame for the Universe during the epoch of last scattering. Given this fact, one may be tempted to construct gravitational theories that have a preferred frame from the outset while preserving diffeomorphism invariance. But what more is there to gain from working with gravitational theories that violate Lorentz invariance?

Recently, some authors have constructed theories of gravity that have preferred-frame effects (i.e., an Einstein aether), but preserve spatial diffeomorphisms. One of the attractive features of a class of these models, namely Hořava-Lifshitz gravity (HL) [1], is that, due to their anisotropic scaling, implementation of standard field theory methods renders the UV behavior of gravity perturbatively finite. Therefore in this scheme, Lorentz invariance can emerge in the IR, but its violation at shorter scales can cure the UV infinities that usually plague perturbative general relativity.

Despite the promise that HL gravity provides, breaking of refoliation invariance has led to certain technical issues, most notably the presence of an extra scalar graviton mode [2]. The theory could certainly be improved with the import of extra ingredients coming from other walks of gravitational theory. It is interesting that the discreteness of space-time in loop quantum gravity (LQG) also provides a natural UV regulator [3] and one is led to wonder if the finiteness in HL gravity is connected to the nonperturbative discreteness found in LQG. A way to begin analyzing this possible connection is to see if HL gravity can be reexpressed in terms of the Ashtekar canonical variables which naturally lead to the holonomy representation of LQG.

In this paper we show that HL gravity can indeed be reexpressed in terms of Ashtekar's variables and a new physical interpretation of the HL theory emerges, which paves a way of understanding a manifestly 4*D* formulation of HL without the need for an extra scalar degree of freedom. What we will discover is that when Dirac fermions couple to the gravitational field via the Ashtekar variables, HL gravity emerges when the spatial fermion current identically vanishes. Vanishing of fermionic currents in equivalent physical systems has been considered e.g., in Refs. [4,5], and we refer to these works for a detailed analysis. For us it is interesting to note that the frame in which this happens supplies the "preferred" foliation of the theory. In Hořava-Lifshitz gravity the finiteness of the graviton arises due to the presence of the Cotton tensor which was assumed. In this work we discover a physical reason for this in the Ashtekar variables: when the condition

[*]salexand@haverford.edu
[†]j.magueijo@imperial.ac.uk
[‡]amarcian@haverford.edu

for the York time [6] is imposed, the extrinsic curvature gets related to the Cotton tensor and the York time is identified with the zeroth component of the fermion current, i.e., the charge density. In this phase, HL gravity has the interpretation of a timelike current (Fermi aether) that fills space-time. We also show an equivalence of the scalar, vector and Gauss constraints between HL gravity and the Ashtekar constraints when the spatial fermion current vanishes.

II. HL THEORY IN ASHTEKAR VARIABLES

One cannot overemphasize the importance of spinors in understanding gravity and its quantization. Starting from Weyl, it was understood that the simplest way to couple spinors to gravity involved the so-called spin connection, in the "Cartan-Palatini" formulation of general relativity. Later Kibble realized that general relativity could be seen as the gauge theory of the Poincaré group, with the tetrad-gauging translations and the spin-connection-gauging Lorentz transformations and rotations. Torsion naturally sneaks into the theory whenever spinors are present, although the relation is purely algebraic, so that torsion can be reinterpreted as a four-fermion interaction in the standard torsionfree theory (for an excellent review see Ref. [7] and reference therein).

To a large extent the Ashtekar formalism is a reformulation of the Palatini-Cartan-Kibble earlier work, rendering it more amenable to quantization via techniques imported from lattice gauge theory. The Ashtekar theory can be obtained by adding a surface term to the usual Palatini action. Depending on how this is done in the spinorial sector, one may end up with the same classical dynamics or with an extension of the original theory when spinors are present, as we shall see in the next section. In either case the quantum theory is always distinct from what one would get by attempting to quantize the original theory. Quantum effects and classical dynamics driven by spinors always introduce novelties.

One may wonder how the HL theory looks using Ashtekar's "new" variables. This is most easily accomplished following the treatment in Ref. [3], where the Ashtekar formalism is derived from the standard Arnowitt-Deser-Misner (ADM) framework by an extension of the phase space followed by a canonical transformation (dependent on the Immirzi parameter γ). The first operation produces a canonical pair made up of the densitized inverse triad E_i^a and the extrinsic curvature one-form K_a^i.[1] With E_i^a the inverse of E_i^a, the extrinsic curvature K_{ab} can be obtained from the "extended" K_a^i according to

$$K_{ab} = \sqrt{q}K_{(a}^i E_{b)}^i, \tag{1}$$

subject to the constraint,

$$G_{ab} = K_{[a}^i E_{b]}^i = 0, \tag{2}$$

(which produces a form of the Gauss constraint when contracted with ϵ^{cab}). A canonical transformation dependent on Immirzi parameter γ is then applied to K_a^i leading to the Ashtekar connection

$$A_a^i = \gamma K_a^i + \Gamma_a^i, \tag{3}$$

where, in the absence of spinors, $\Gamma_a^i = \tilde{\Gamma}_a^i$ is the torsionfree Cartan connection associated with E_i^a. The Gauss constraint implies $D_a E_i^a = \partial_a E_i^a + \epsilon_{ijk}\Gamma_a^j E_k^a = 0$, which leads to an expression in terms of the new covariant derivative

$$\mathcal{G}_i = \mathcal{D}_a E_i^a = \partial_a E_i^a + \epsilon_{ijk}A_a^j E_k^a = 0. \tag{4}$$

This is the usual form for the Gauss constraint in terms of Ashtekar variables. The Gauss constraint is the only new constraint to be added in this approach to the usual two present in the ADM formalism.

Having performed this exercise, the ADM Hamiltonian becomes the sum of three constraints: the Gauss, the diffeomorphism and the Hamiltonian constraint. Specifically the Hamiltonian constraint becomes

$$\mathcal{H}_{\text{Ash}} = \frac{1}{2\kappa\sqrt{q}}E_i^a E_j^b(\epsilon^{ij}{}_k F_{ab}^k - 2(\gamma^2 + 1)K_{[a}^i K_{b]}^j), \tag{5}$$

where we are using units such that $\kappa = 8\pi G$.

We now note that the HL action can be written as the standard Einstein-Hilbert action plus an additional term in $1 - \lambda$

$$S_{\text{HL}} = S_{\text{EH}} + \frac{1-\lambda}{2\kappa}\int d^3x dt \sqrt{q}NK^2. \tag{6}$$

This results in a correction to the ADM Hamiltonian

$$\mathcal{H}_{\text{HL}} = \mathcal{H}_{\text{ADM}} + \frac{\sqrt{q}}{2\kappa}(\lambda - 1)K^2. \tag{7}$$

Therefore all we need to do in order to translate the model into the Ashtekar formalism is to rewrite the extra term in terms of the canonically transformed variables. It is easy to prove that

$$K = q^{ab}K_{ab} = \frac{1}{\sqrt{q}}E_i^a K_a^i, \tag{8}$$

so that the Hamiltonian constraint becomes

$$\mathcal{H}_{\text{HL}} = \frac{1}{2\kappa\sqrt{q}}E_i^a E_j^b(\epsilon^{ij}{}_k F_{ab}^k - 2(\gamma^2 + 1)K_{[a}^i K_{b]}^j$$
$$+ (1 - \lambda)K_a^i K_b^j). \tag{9}$$

We see that the diffeomorphism-invariant theory contains both the trace and the traceless part in well-apportioned amounts. The new term is a pure trace, deforming the

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[1]From now on, we will label space indices with latin letters a, b, with a, $b = 1, 2, 3$, and internal SU(2) indices with latin letters i, j, with i, $j = 1, 2, 3$.

original proportions. Notice finally that when we select the values $\lambda = 1 + 2(\gamma^2 + 1)$, the theory becomes

$$\mathcal{H}_{\text{HL}} = \frac{1}{2\kappa\sqrt{q}} E_i^a E_j^b (\epsilon^{ij}{}_k F_{ab}^k + 2(1 - \lambda) K_{[a}^i K_{b]}^j + (1 - \lambda) K_{(a}^i K_{b)}^j). \tag{10}$$

Our task now is to obtain this theory from a fermionic aether. In so doing it will be useful to recall that in the above Hamiltonian K_a^i is to be understood as

$$K_a^i = \frac{A_a^i - \Gamma_a^i}{\gamma}. \tag{11}$$

Thus, if Γ_a^i acquires torsion (solved explicitly in terms of the fermionic field), it is not unreasonable to expect that a new term, of the form of the new term in $(1 - \lambda)$, is generated.

III. EINSTEIN-HILBERT ACTION AND COUPLING TO MASSLESS FERMIONS

A direct way to see how HL gravity is related to the Ashtekar variables is to consider a 4D gravitational Holst action in the first-order formalism which can naturally be reduced to the Ashtekar variables [8–12]

$$S_{\text{EHC}}(e, A, \psi, \overline{\psi})$$
$$= \frac{1}{2\kappa} \int_{\mathcal{M}} \left(\frac{\epsilon_{IJKL}}{2} e^I \wedge e^J \wedge F^{KL} - \frac{1}{\gamma} e_I \wedge e_J \wedge F^{IJ} \right)$$
$$+ \frac{i}{2} \int_{\mathcal{M}} \star e_I \wedge \left[\overline{\psi} \gamma^I \left(1 - \frac{i}{\alpha} \gamma_5 \right) \mathcal{D}\psi \right.$$
$$\left. - \overline{\mathcal{D}\psi} \left(1 - \frac{i}{\alpha} \gamma_5 \right) \gamma^I \psi \right], \tag{12}$$

in which antisymmetrized pairs AB, with $A, B = 0, 1, 2, 3$, are internal indices of the adjoint representation of $\mathfrak{so}(3, 1)$ and the symbol $\mathcal{D}$ denotes covariant derivative with respect to the $SO(3, 1)$ connection A^{IJ}, the field strength of which is R^{IJ}. Notice that this action differs from the one considered in Ref. [8] by an axial coupling in the fermionic term. It was shown in Ref. [9] that this action is equivalent to the Einstein-Cartan action at the effective level. We can immediately identify the Ashtekar-Barbero connection as a spatial projection of the spin connection

$$A_b^{\prime j} \equiv -\gamma A_b{}^{j0} - \frac{1}{2} \epsilon^j{}_{kl} A_b{}^{kl} = \gamma K_b^j + \Gamma_b^j, \tag{13}$$

where both sets of indices run from 1 to 3, as previously stated. The remaining components of the space-time connection A are recast into

$$-A_b^{\prime j} \equiv A_b{}^{j0} - \frac{1}{2\gamma} \epsilon^j{}_{kl} A_b{}^{kl}. \tag{14}$$

Finally the components A_t^{IJ} are nondynamical, as are the lapse function N and shift vector N^a appearing in the metric. Variation with respect to the nondynamical connection

components gives partially second-class constraints. These constraints can be solved, giving the results

$$\gamma^- A_b^{\prime k} = -A_b^{\prime k} + 2\Gamma_b^k. \tag{15}$$

Following [12], we rewrite the connection Γ_b^k as

$$\Gamma_b^k = \tilde{\Gamma}_b^k + \frac{\gamma\kappa}{4(1 + \gamma^2)} (\theta \epsilon_{ij}{}^k e_b^i \mathcal{J}^j - \beta e_b^k \mathcal{J}^0), \tag{16}$$

i.e., the sum of the metric compatible spin connection $\tilde{\Gamma}_b^k$ and a torsion contribution

$$C_a^j \equiv \frac{\gamma\kappa}{4(1 + \gamma^2)} (\theta \epsilon^j{}_{kl} e_a^k \mathcal{J}^l - \beta e_a^j \mathcal{J}^0), \tag{17}$$

with coefficients

$$\beta = \gamma + \frac{1}{\alpha} \quad \text{and} \quad \theta = 1 - \frac{\gamma}{\alpha}, \tag{18}$$

where the currents are defined as

$$\mathcal{J}^0 = \phi^\dagger\phi - \chi^\dagger\chi, \qquad \mathcal{J}^i = \phi^\dagger\sigma^i\phi + \chi^\dagger\sigma^i\chi, \tag{19}$$

in terms of the spin components $\psi = (\phi, \chi)^T$. Furthermore, A_t^{k0} is determined by another second-class constraint, requiring $\epsilon_{ijk} A_t^{jk}$ to remain free as Lagrange multiplier of the Gauss constraint.

With the definitions above the Gauss constraint becomes

$$\mathcal{G}_i = \gamma[K_b, E^b]_i - \frac{\gamma\beta}{2(1 + \gamma^2)} \sqrt{q} \mathcal{J}_i. \tag{20}$$

The diffeomorphism constraint reads:

$$\mathcal{C}_a = \frac{1}{\gamma} E_j^b F_{ab}^j - \frac{i}{\gamma} \sqrt{q} (\theta_L(\phi^\dagger D_a\phi - \overline{D_a\chi}\chi) - \text{c.c.})$$
$$+ -\frac{\gamma^2 + 1}{\gamma^2} K_a^j G_j, \tag{21}$$

while the Hamiltonian constraint is:

$$\mathcal{C} = \frac{1}{2\kappa\sqrt{q}} E_i^a E_j^b (\epsilon^{ij}{}_k F_{ab}^k - 2(\gamma^2 + 1) K_{[a}^i K_{b]}^j)$$
$$+ \frac{\beta}{2\kappa\gamma\sqrt{q}} E_i^a \Delta_a (\sqrt{q} \mathcal{J}^i) + (1 + \gamma^2)\kappa\tilde{D}_a \left(\frac{E_i^a G^i}{\sqrt{q}} \right)$$
$$+ \frac{i}{\gamma\kappa} E_i^a (\theta_L(\phi^\dagger\sigma^i\Delta_a\phi + \overline{\Delta_a\chi}\sigma^i\chi) + -\theta_R(\chi^\dagger\sigma^i\Delta_a\chi$$
$$+ \overline{\Delta_a\phi}\sigma^i\phi)) + \frac{1}{4\kappa\gamma^2} \left(3 - \frac{\gamma}{\alpha} + 2\gamma^2 \right) \epsilon_{lkr} K_a^l E_k^a \mathcal{J}^r, \tag{22}$$

where D is the covariant derivative with respect to Γ_b^k, $\tilde{D}$ is the covariant derivative with respect to compatible connection $\tilde{\Gamma}_b^k$, and we have introduced $\theta_{L/R} \equiv \frac{1}{2}(1 \pm i/\alpha)$. The derivative Δ stands for the covariant derivative related to the "corrected connection" $\mathcal{A}_a^i$ (see Ref. [12] for a detailed description), whose expression in terms of the connection $A_a^i = \tilde{A}_a^i + \bar{A}_a^i$ accounting for the torsionfull components $\bar{A}_a^i$ is given by

$$\mathcal{A}_a^{\ i} \equiv A_a^i + \frac{\gamma\kappa}{4\alpha}e_a^i\mathcal{J}^0, \qquad (23)$$

where $\tilde{A}_a^i = A'^i_{\ a}$. In the notation of [12] the Ashtekar-Barbero connection splits into a torsion part and a torsion-free part. Specifically, with $\tilde{\Gamma}_a^i$ the compatible torsionfree spin connection and $\tilde{K}_a^i$ the compatible torsionfree extrinsic curvature, we have

$$A_a^i = \tilde{\Gamma}_a^i + \gamma\tilde{K}_a^i + \frac{\kappa\gamma}{4}\epsilon^i_{\ kl}e_a^k\mathcal{J}^l - \frac{\kappa\gamma}{4\alpha}e_a^i\mathcal{J}^0. \qquad (24)$$

The three constraints (20)–(22) provide a set of first-class constraints.

IV. NONMINIMAL ECH ACTION IN METRIC-COMPATIBLE VARIABLES

We focus on the term in $\mathcal{C}$, as its generalization introduces us to the Hamiltonian formulation of the Hořava-Lifshitz dynamics. The Gauss and the vector constraints of the Einstein-Cartan-Holst (ECH) action will indeed close weakly on the constraints' surfaces, the same constraints' algebra where the Hořava-Lifshitz theory of gravity (6) closes, provided some extra conditions are satisfied. In contrast, the Hořava-Lifshitz term in (6), in the scalar constraint, $\mathcal{H}_{\text{HL}}$, endows the De Witt metric with a conformal dimensionless coupling λ. For $\lambda < 1/3$ gravity becomes repulsive, and it is interesting to notice that this condition corresponds to a region in the plane spanned by the Immirzi parameter γ and nonminimal Fermion coupling constant α.

We start from the scalar constraint for the Einstein-Cartan-Holst action (12) recast in terms of the "metric compatible" Ashtekar variables

$$\mathcal{H}_{\text{Ash}}^{\text{ECH}} = \frac{1}{2\kappa\sqrt{q}}E_i^a E_j^b(\epsilon^{ij}_{\ k}F_{ab}^k - 2(\gamma^2+1)K_{[a}^i K_{b]}^j)$$

$$+ \frac{i}{2\gamma}E_i^a(\phi^\dagger\sigma^i\partial_a\phi - \chi^\dagger\sigma^i\partial_a\chi - \text{c.c.})$$

$$+ \frac{\theta}{2\gamma}E_j^b\tilde{\Gamma}_b^j\mathcal{J}^0 + \frac{\gamma}{4\alpha\sqrt{q}}\epsilon^{ij}_{\ k}E_i^a e_b^k\mathcal{J}^0\partial_a E_j^b$$

$$+ \frac{3\kappa}{16}\frac{\sqrt{q}}{1+\gamma^2}\left(\frac{1}{\alpha^2} - \frac{2}{\alpha\gamma} - 1\right)(\mathcal{J}_0^2 - \mathcal{J}_l\mathcal{J}^l)$$

$$+ \frac{1}{\kappa\gamma^2}\tilde{D}_a\left(\frac{E_i^a\tilde{G}^i}{\sqrt{q}}\right) + \frac{2+\gamma^2}{4\gamma^2}\tilde{G}_i\mathcal{J}^i, \qquad (25)$$

where the tilde "~" labels metric-compatible quantities. We will show that it is possible to reduce $\mathcal{H}_{\text{Ash}}^{\text{ECH}}$ to the Hořava-Lifshitz gravity scalar constraint

$$\mathcal{H}_{\text{HL}} = \frac{1}{2\kappa\sqrt{q}}E_i^a E_j^b(\epsilon^{ij}_{\ k}F_{ab}^k - 2(\gamma^2+1)K_{[a}^i K_{b]}^j$$

$$+ (1-\lambda)K_a^i K_b^j), \qquad (26)$$

by assuming some restrictions on the quantum states of the Fermionic matter content of (12). In order to show the equivalence of the two theories, we must also check that the vector constraint $\mathcal{C}_a$ and the Gauss constraint $\mathcal{G}_i$, once recast in the metric-compatible variables, reduce to the ones of the Hořava-Lifshitz theory provided some assumptions (that will soon be listed) are fulfilled.

We first rewrite the Gauss and vector constraints in terms of metric-compatible quantities. In the presence of fermions the Gauss constraint is modified to

$$\mathcal{G}_i = D_b E_i^b - \frac{1}{2}\sqrt{q}\mathcal{J}_i = \gamma[K_b, E^b]_i - \frac{\gamma\beta}{2(1+\gamma^2)}\sqrt{q}\mathcal{J}_i. \qquad (27)$$

We see that when the spatial current vanishes ($\mathcal{J}_l = 0$) the Gauss constraint reduces to $\mathcal{G}_i = \gamma\epsilon^k_{\ ji}K_b^k E_j^b = \gamma\epsilon^k_{\ ji}\tilde{K}_b^k E_j^b = \tilde{\mathcal{G}}_i$. We can express the vector constraint in terms of metric-compatible variables as

$$\mathcal{C}_a = \frac{1}{\gamma}E_j^b\tilde{D}_{[a}\tilde{K}_{b]}^j + \text{sign}(\det e_a^i)\frac{\kappa}{4}\epsilon_{ca}^{\ \ b}E_l^c\tilde{D}_b(\sqrt{q}\mathcal{J}^l)$$

$$+ -\frac{i}{2\gamma}\sqrt{q}(\phi^\dagger\tilde{D}_a\phi + \chi^\dagger\tilde{D}_a\chi - \text{c.c.})$$

$$+ \frac{1}{\gamma}\text{sign}(\det e_a^i)E_l^d(\epsilon_{cd}^{\ \ b}\Gamma_{ba}^c - \epsilon_{ca}^{\ \ b}\Gamma_{bd}^c)\sqrt{q}\mathcal{J}^l$$

$$+ \left(\frac{\kappa}{4}\epsilon^{jkl}\mathcal{J}_k e_{al} - \frac{\kappa}{4\alpha}e_a^j\mathcal{J}^0 - \frac{1+\gamma^2}{\gamma}K_a^j\right)\tilde{\mathcal{G}}_j, \qquad (28)$$

which reduces to the expression

$$\mathcal{C}_a = \frac{1}{\gamma}E_j^b\tilde{D}_{[a}\tilde{K}_{b]}^j - \left(\frac{\kappa}{4\alpha}e_a^j\mathcal{J}^0 + \frac{1+\gamma^2}{\gamma}K_a^j\right)\tilde{\mathcal{G}}_j$$

$$= \tilde{\mathcal{C}}_a - \left(\frac{\kappa}{4\alpha}e_a^j\mathcal{J}^0 + \frac{1+\gamma^2-\gamma^3}{\gamma}K_a^j\right)\tilde{\mathcal{G}}_j \qquad (29)$$

on states over which $\mathcal{J}^l$ and $(\phi^\dagger\tilde{D}_a\phi + \chi^\dagger\tilde{D}_a\chi - \text{c.c.})$ vanish. Once we classically implement $\mathcal{G}_i = \tilde{\mathcal{G}}_i = 0$, it follows that the vector constraint of the Einstein-Cartan-Holst theory (12) becomes equivalent to the one expressed in terms of the metric compatible variables in Hořava-Lifshitz gravity. For this to be true it is essential that the spatial currents remain switched off (for a discussion of this condition see e.g., Refs. [4,5]).

A. Fixed values of the Immirzi parameter and λ

In this section we explore the case $\theta = 0$, emphasizing that the equality it encodes between the two parameters entering the Einstein-Cartan-Holst action (i.e., $\alpha = \gamma$) is not necessarily required. Indeed α does not need to be fixed to γ for an equivalence between the Hořava-Lifshitz theory of gravity endowed with the square of the Cotton tensor, namely the term $C_{ij}C^{ij}$, and the action in (12) to be found. Nevertheless, we start from this instructive and simple case. Throughout this section, we assume that[2] $\langle(\phi^\dagger\tilde{D}_a\phi + \chi^\dagger\tilde{D}_a\chi - \text{c.c.})\rangle$ and $\langle\mathcal{J}_i\rangle$ vanish, as a necessary condition

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[2]We denote with "$\langle\cdot\rangle$" the expectation value of operators on the quantum state realizing our assumptions.

for our claim. We then recast the theory in terms of torsion-full Ashtekar variables $\{A_a^i, E_j^b\}$, which in turn are given, in terms of the compatible variables $\{\tilde{A}_a^i, \tilde{E}_j^b\}$, by

$$A_a^i = \tilde{A}_a^i - \frac{\gamma\kappa}{4\alpha} e_a^i \mathcal{J}^0, \qquad E_j^b = \tilde{E}_j^b. \qquad (30)$$

This relation yields general extrinsic curvature, K_a^i, which has a torsionfree part, $\tilde{K}_a^i$, and a torsionfull piece $\bar{K}_a^i$

$$K_a^i = \tilde{K}_a^i - \frac{\kappa}{4\alpha} e_a^i \mathcal{J}^0 \equiv \tilde{K}_a^i + \bar{K}_a^i, \qquad (31)$$

where the torsionfull extrinsic curvature is $\bar{K}_a^i = -\frac{\kappa}{4\alpha} e_a^i \mathcal{J}^0$.

As we are rewriting our theory in terms of torsionfull quantities, for the sake of consistency the field strength must be expressed in terms of the torsionfull connection A_a^i. It is not difficult to check that the scalar constraint (25) rewrites as

$$\begin{aligned} \mathcal{H}_{\text{Ash}}^{\text{ECH}} &= \frac{1}{2\kappa\sqrt{q}} E_i^a E_j^b (\epsilon^{ij}_{\ k}(F_{ab}^k - 2(\gamma^2+1)K_{[a}^i K_{b]}^j) \\ &\quad + \frac{i}{2\kappa\gamma} E_i^a (\phi^\dagger \sigma^i \tilde{D}_a \phi - \chi^\dagger \sigma^i \tilde{D}_a \chi - \text{c.c.}) \\ &\quad + \frac{E_i^a}{2\kappa\sqrt{q}} \tilde{D}_a(\sqrt{q}\mathcal{J}^i) + \frac{1}{2\kappa} E_j^b K_b^j \mathcal{J}^0 \\ &\quad + \frac{1}{2\kappa\gamma}[K_a, E^a]_j \mathcal{J}^j + -\frac{3}{8\kappa\sqrt{q}}\frac{1}{1+\gamma^2}q\mathcal{J}_0^2 \\ &\quad + \frac{1+\gamma^2}{\kappa\gamma^2} \tilde{D}_a\left(\frac{E_i^a \mathcal{G}^i}{\sqrt{q}}\right). \end{aligned} \qquad (32)$$

Provided now that $\langle \mathcal{J}^i \rangle = \langle (\phi^\dagger \sigma^i \tilde{D}_a \phi - \chi^\dagger \sigma^i \tilde{D}_a \chi - \text{c.c.}) \rangle = 0$, on the constraint's surface where $\mathcal{G}^i = 0$ the scalar constraint (32) written in terms of torsionfull quantities reads

$$\begin{aligned} \mathcal{H}_{\text{Ash}}^{\text{ECH}} &= \frac{1}{2\kappa\sqrt{q}} E_i^a E_j^b (\epsilon^{ij}_{\ k} F_{ab}^k - 2(\gamma^2+1)K_{[a}^i K_{b]}^j) \\ &\quad + -\frac{3}{8\kappa\sqrt{q}}\frac{1}{1+\gamma^2}q\mathcal{J}_0^2. \end{aligned} \qquad (33)$$

A few algebraic manipulations are now in order. Firstly note that

$$\frac{3\kappa}{16}\frac{\sqrt{q}}{\gamma^2}\mathcal{J}_0^2 = \frac{2}{3}\frac{E_i^a E_j^b}{2\kappa\sqrt{q}}\left(\frac{\kappa}{4\gamma}e_{(a}^i \frac{\kappa}{4\gamma}e_{b)}^j\right)\mathcal{J}_0^2, \qquad (34)$$

in which the symmetrization arises from the fact that

$$\begin{aligned} 9e^2 &= (E_i^a e_a^i)(E_j^b e_b^j) = E_i^a E_j^b (e_{(a}^i e_{b)}^j + e_{[a}^i e_{b]}^j) \\ &= E_i^a E_j^b e_{(a}^i e_{b)}^j, \end{aligned}$$

where symmetrization and skew symmetrization are intended to have been normalized (recall too that $\sqrt{q} = e$). It is then straightforward to recognize that

$$\frac{2}{3}\frac{E_i^a E_j^b}{2\kappa\sqrt{q}}\left(\frac{\kappa}{4\gamma}e_{(a}^i \frac{\kappa}{4\gamma}e_{b)}^j\right)\mathcal{J}_0^2 = \frac{2}{3}\frac{E_i^a E_j^b}{2\kappa\sqrt{q}}\bar{K}_{(a}^i \bar{K}_{b)}^i \qquad (35)$$

and that

$$\begin{aligned} &\frac{2}{3}\frac{E_i^a E_j^b}{2\kappa\sqrt{q}}\left(\frac{\kappa}{4\gamma}e_a^i \frac{\kappa}{4\gamma}e_b^j\right)\mathcal{J}_0^2 \\ &= \frac{2}{3}\frac{E_i^a E_j^b}{2\kappa\sqrt{q}}(K_a^i K_b^j + \tilde{K}_a^i \tilde{K}_b^j - 2K_a^i \tilde{K}_b^j), \end{aligned} \qquad (36)$$

having made use of the definition of $\bar{K}_a^i$ in (31) and again the identities $E_i^a e_a^i = 3e$ and $E_i^a = ee_i^a$. If we impose that the trace of the extrinsic curvature vanishes, $K = 0$, which in terms of metric-compatible variables, is equivalent to imposing

$$\tilde{K} = -\frac{3}{4\gamma}\mathcal{J}_0, \qquad (37)$$

we obtain

$$\frac{3\kappa}{16}\frac{\sqrt{q}}{\gamma^2}\mathcal{J}_0^2 = \frac{2}{3}\frac{E_i^a E_j^b}{2\kappa\sqrt{q}}(K_a^i K_b^j + \tilde{K}_a^i \tilde{K}_b^j). \qquad (38)$$

We emphasize that condition (37) (which generalizes the Lichnerowicz condition $\tilde{K} = 0$) corresponds to the second-class constraint imposed to the ADM formulation of gravity $\Pi = \mathcal{Y}$ (Π^{ab} being the conjugate momentum to q_{ab}) while solving the vector and scalar constraints. The York time[3] $\mathcal{Y}$ is then identified with the fermionic electric charge density:

$$\mathcal{Y} = -\frac{3}{4\gamma}\mathcal{J}_0. \qquad (39)$$

Once these algebraic manipulations are considered, it immediately follows that

$$\begin{aligned} \mathcal{H}_{\text{Ash}}^{\text{ECH}} &= \frac{1}{2\kappa\sqrt{q}} E_i^a E_j^b\left(\epsilon^{ij}_{\ k} F_{ab}^k - 2(\gamma^2+1)K_{[a}^i K_{b]}^j \right. \\ &\quad \left. + -\frac{2}{3}\frac{\gamma^2}{1+\gamma^2}\bar{K}_{(a}^i \bar{K}_{b)}^i\right). \end{aligned} \qquad (40)$$

Therefore, when $\lambda = 3 + 2\gamma^2$ and $3(\gamma^2+1)^2 = \gamma^2$, which respectively fix the values $\gamma^2 = \{-3, -1/3\}$ and $\lambda = \{-3, 7/3\}$, we find that the scalar constraint for action (12) is equivalent to the scalar constraint of the Hořava-Lifshitz gravity theory, i.e.,

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[3]More precisely, we should consider the definition of the York time provided in Ref. [6] (and recalled in Ref. [13]), in which the trace of Π^{ab} is rescaled by the inverse of $\sqrt{q}$ in order to provide a variable canonically conjugated to the Hamiltonian density $\sqrt{q}$. It is not probably surprising the fact that this automatically encodes a treatment of fermionic matter in terms of densitized fields (see e.g., Refs. [3,14,15]).

$$\mathcal{H}_{\text{Ash}}^{\text{ECH}} = \frac{1}{2\kappa\sqrt{q}} E_i^a E_j^b (\epsilon^{ij}{}_k F_{ab}^k - 2(\gamma^2 + 1) K_{[a}^i K_{b]}^j) + (1 - \lambda) K_a^i K_b^j),$$

(41)

with

$$\lambda = 3 + 2\gamma^2 = \{-3, 7/3\}.$$

(42)

In terms of the torsionfull variables, the Gauss and the vector constraint becomes

$$\mathcal{G}_i = D_b E_i^b - \frac{1}{2}\sqrt{q}\mathcal{J}_i = \gamma[K_b, E^b]_i - \frac{\gamma\beta}{2(1 + \gamma^2)}\sqrt{q}\mathcal{J}_i$$

(43)

and

$$\mathcal{C}_a = \frac{1}{\gamma} E_j^b F_{ab}^j - \frac{1 + \gamma^2}{\gamma^2} K_a^i - \frac{i}{2\gamma}\sqrt{q}(\phi^\dagger \tilde{D}_a \phi + \chi^\dagger \tilde{D}_a \chi)$$
$$+ -\frac{\kappa\sqrt{q}}{8(1 + \gamma^2)}(\theta\epsilon^j{}_{kl} e_a^k \mathcal{J}^l - \beta e_a^j \mathcal{J}_0)\mathcal{J}_i - \frac{1}{2} K_a^i \sqrt{q}\mathcal{J}_i.$$

(44)

Again, under the assumptions $\langle\mathcal{J}^i\rangle = \langle(\phi^\dagger \sigma^i \tilde{D}_a \phi - \chi^\dagger \sigma^i \tilde{D}_a \chi - \text{c.c.})\rangle = 0$, we recover that $\mathcal{G}_i$ and $\mathcal{C}_a$ have the same form as the Gauss and the vector constraints of the Hořava-Lifshitz theory of gravity, provided that torsionfree quantities are replaced everywhere by torsionfull quantities.

It is remarkable that when the York-time condition is imposed, $K = 0$, the Cotton tensor is naturally present in the theory. Indeed, as shown in Ref. [16] using Ashtekar variables, under the assumption $K = 0$ the constraints imply

$$\tilde{K}^{ab} = k\varepsilon^{abd}\tilde{D}_a\left(\tilde{R}_d{}^b - \frac{1}{4}\delta_d^b\tilde{R}\right) = k\tilde{C}^{ab},$$

(45)

with $\tilde{R}_a{}^b$ the three-dimensional Ricci tensor and $\tilde{R}$ its contraction, ε^{abd} the Levi-Civita tensor $\varepsilon^{abd} = \epsilon^{ijk} e_i^a e_j^b e_k^c$, k a constant of proportionality and $\tilde{C}^{ab}$ the Cotton tensor in 3D in terms of metric-compatible variables. We recall that the action for the $z = 3$ Hořava-Lifshitz theory of gravity in 3 + 1D takes the form

$$S^{\text{HL}} = \int dt d^3x \sqrt{q} N\left(\frac{2}{\kappa^2}(\tilde{K}_{ij}\tilde{K}^{ij} - \lambda\tilde{K}^2) - \frac{\kappa^2}{2w^4}\tilde{C}_{ij}\tilde{C}^{ij}\right),$$

(46)

which after Wick rotation to imaginary time may be rewritten as a sum of squares

$$S^{\text{HL}} = 2i \int dt d^3x \sqrt{q} N\left(\frac{1}{\kappa}\tilde{K}_{ij} - \frac{\kappa}{2w^2}\tilde{C}_{ij}\right) G^{ijkl}$$
$$\times \left(\frac{1}{\kappa}\tilde{K}_{kl} - \frac{\kappa}{2w^2}\tilde{C}_{kl}\right),$$

(47)

where we have introduced the de Witt metric

$$G^{ijkl} = \frac{1}{2}(q^{ik}q^{jl} + q^{il}q^{jk}) - \lambda q^{ij}q^{kl}.$$

(48)

When we impose (37), we find that (45) relates the metric-compatible extrinsic curvature and the metric-compatible Cotton tensor. Therefore, the two tensors depend on the extrinsic curvature terms that only appear in the scalar constraint (41) and in action (6). This would finally account for recovering an action similar in form to (46) after having properly Wick-rotated the time coordinate, but with contribution originated by the presence of fermions (and consequently of torsion). On shell, for solutions of the Hamiltonian constraints derived from (12) once (37) is imposed, a relation similar in form to (6) can be recovered, but now in terms of torsionfull quantities. The Wick-rotated action is then

$$S = 2i \int dt d^3x \sqrt{q} N\left[\frac{1}{\kappa}(\tilde{K}_{ij} + \bar{K}_{ij})G^{ijkl}\frac{1}{\kappa}(\tilde{K}_{kl} + \bar{K}_{kl})\right].$$

By introducing real parameter ξ and using $\tilde{K}^{ab} = k\tilde{C}^{ab}$ we can write the Euclidean action as

$$S = 2i \int dt d^3x \sqrt{q} N$$
$$\times \left\{\left[\frac{1 - \xi}{\kappa}(\tilde{K}_{ij} + \bar{K}_{ij})G^{ijkl} \times \frac{1 - \xi}{\kappa}(\tilde{K}_{kl} + \bar{K}_{kl})\right]\right.$$
$$\left. + \frac{\xi^2}{\kappa^2}\tilde{C}_{ij}\tilde{C}^{ij} + 2\xi\tilde{K}_{ij}G^{ijkl}\bar{K}_{kl}\right\}.$$

This finally becomes

$$S = 2i \int dt d^3x \sqrt{q} N\left\{\frac{1}{\kappa'^2} K_{ij}G^{ijkl}K_{ij} + \frac{\xi^2}{(1 - \xi)^2\kappa'^2}\tilde{C}_{ij}\tilde{C}^{ij}\right.$$
$$\left. - 2\frac{\xi^2}{(1 - \xi)^2\kappa'^2}\bar{K}_{ij}G^{ijkl}\bar{K}_{kl}\right\},$$

once we recognize that $\tilde{K}_{ij}G^{ijkl}\tilde{C}_k l$ can be written as a total derivative [1], for parameters

$$\frac{\xi^2 k^2}{(1 - \xi)^2} = \frac{\kappa'^4}{2w^2} \quad \text{and} \quad (\gamma^2, \lambda) = \left\{(-3, -3), \left(-\frac{1}{3}, \frac{7}{3}\right)\right\},$$

(49)

and absorbing $1 - \xi$ in κ, by defining

$$\kappa' = \kappa/(1 - \xi).$$

At the end of this procedure, we obtain the Euclidean action

$$S = 2i \int dt d^3x \sqrt{q} N\left\{\frac{1}{\kappa'^2} K_{ij}G^{ijkl}K_{ij} + \frac{\kappa'^2}{2w^2}\tilde{C}_{ij}\tilde{C}^{ij}\right.$$
$$\left. - \frac{(1 - 3\lambda)}{\gamma^2}\frac{\kappa'^4}{w^2}\frac{3(1 - \xi)^2}{16}\mathcal{J}_0^2\right\}.$$

(50)

Action (50) contains a $\mathcal{J}_0^2$ interaction-term additional to the action for the Hořava-Lifshitz gravity in Ref. [1]. Without fixing λ the last term in (50) can be made to

vanish for the degenerate value $\lambda = 1/3$ [instead of (49)]. By properly dealing with α more general solutions can be found, and the conditions relating γ and λ and the vanishing of the $\mathcal{J}_0^2$ terms can be met simultaneously. We will revisit this issue in the next subsection. We close this section with a remark on the two possible values of γ^2, and hence λ, which we have found were needed for our equivalence. It is well known [17] that the physical meaning of λ can be inferred from the analysis of the acceleration of the three-volume $V \equiv \int d^3x \sqrt{q}$, which is encoded in the formula

$$\frac{d^2}{dt^2} V = -\frac{2}{3\lambda - 1} \int d^3x \sqrt{q}\tilde{R}.$$ (51)

Therefore, an attractive gravitational force is recovered for $\gamma^2 = -1/3$ and $\lambda = 7/3$ in this framework.

B. The of α parameter solutions in HL gravity

In this subsection we show how it is possible to extend our results, by dropping the constraint $\alpha = \gamma$ and leading to a one-parameter family of solutions in λ and γ depending on the nonminimal coupling α entering the Einstein-Cartan-Holst action (12). We will show that it is possible to impose the vanishing of the extra interaction term in (50) even for $\alpha \neq \gamma$. The scalar constraint will still be given by

$$\mathcal{H}_{\mathrm{Ash}}^{\mathrm{ECH}} = \frac{1}{2\kappa\sqrt{q}} E_i^a E_j^b (\epsilon^{ij}{}_k (F_{ab}^k - 2(\gamma^2 + 1) K_{[a}^i K_{b]}^j)$$

$$+ \frac{i}{2\kappa\gamma} E_i^a (\phi^\dagger \sigma^i \tilde{D}_a \phi - \chi^\dagger \sigma^i \tilde{D}_a \chi - \mathrm{c.c.})$$

$$+ \frac{E_i^a}{2\kappa\sqrt{q}} \tilde{D}_a(\sqrt{q}\mathcal{J}^i) + \frac{1}{2\kappa} E_j^b K_b^j \mathcal{J}^0$$

$$+ \frac{1}{2\kappa\gamma} [K_a, E^a]_j \mathcal{J}^j - \frac{3}{8\kappa\sqrt{q}} \frac{1}{1+\gamma^2} q\mathcal{J}_0^2$$

$$+ \frac{1+\gamma^2}{\kappa\gamma^2} \tilde{D}_a \left(\frac{E_i^a \mathcal{G}^i}{\sqrt{q}}\right),$$ (52)

but now the definition of the torsionfull part of the extrinsic curvature $\bar{K}_a^i = -\frac{\kappa}{4\alpha} e_a^i \mathcal{J}^0$ allows us to reexpress the scalar constraint as

$$\mathcal{H}_{\mathrm{Ash}}^{\mathrm{ECH}} = \frac{1}{2\kappa\sqrt{q}} E_i^a E_j^b \left(\epsilon^{ij}{}_k F_{ab}^k - 2(\gamma^2 + 1) K_{[a}^i K_{b]}^j\right.$$

$$\left. + -\frac{2}{3}\frac{\alpha^2}{1+\gamma^2} \bar{K}_{(a}^i \bar{K}_{b)}^j\right),$$ (53)

in which again we have assumed $\langle \mathcal{J}^i \rangle = \langle (\phi^\dagger \sigma^i \tilde{D}_a \phi - \chi^\dagger \sigma^i \tilde{D}_a \chi - \mathrm{c.c.}) \rangle = 0$. The conditions imposed in order to recover the Hořava-Lifshitz scalar constraint are now $\lambda = 3 + 2\gamma^2$ and $3(\gamma^2 + 1)^2 = \alpha^2$. Therefore the Immirzi parameter and λ are now parametrized by the nonminimal coupling parameter α according to

$$\gamma^2 = \pm\frac{\alpha}{\sqrt{3}} - 1, \quad \text{and} \quad \lambda = 1 \pm \frac{2\alpha}{\sqrt{3}}.$$ (54)

As a consequence, the condition to obtain the degenerate value $\lambda = 1/3$, in order to derive exactly the quadratic Hořava-Lifshitz action in the Euclidean space,

$$S = 2i \int dt d^3x \sqrt{q} N \left\{\frac{1}{\kappa'^2} K_{ij} G^{ijkl} K_{ij} + \frac{\kappa'^2}{2w^2} \tilde{C}_{ij}\tilde{C}^{ij}\right\},$$ (55)

can now be imposed, leading to

$$\alpha = \mp\frac{1}{\sqrt{3}}.$$ (56)

This result sheds new light on the physical meaning of the dimensionless conformal coupling parameter λ, showing its connection with the nonminimal coupling parameter α that appears in (12). It is also interesting to note that any value $\lambda < 3$ implies that only imaginary values are recovered for the Immirzi parameter.

V. CONCLUSIONS

In this paper we have shown how HL theory may be seen, in some situations, as the action of a fermionic aether in Ashtekar-like gravity in the presence of chiral spinor couplings. The torsion induced by the spinor generates an extra term identical to that used in HL theory to break refoliation invariance. This realization of Hořava gravity in the Ashtekar variables clarifies some open questions that were present in the metric-variable formulation. All of these issues are naturally connected by the condition of having a York time, namely that the trace of the extrinsic curvature vanishes. Once this condition is imposed the finiteness of the graviton is understood, since the Cotton tensor, which was assumed in the original Hořava formulation, gets related to the traceless part of the extrinsic curvature. Furthermore, from the vanishing of the trace of the extrinsic curvature, we get a physical interpretation for the York time [6,18,19] as the fermionic electric charge density. This identification can help us understand the issue of the loss of refoliation invariance as the physical fermionic aether which is the York time, an issue we intend to pursue in future work.

Given our results we can speculate further on why anisotropic scaling seems to lead to a renormalizable theory. The Einstein-Cartan-Kibble formulation of gravity is a gravity theory with torsion, but it is in fact equivalent to the torsionfree Einstein-Hilbert formulation if a four-fermion (axial-axial) interaction is added to the latter. It is well-known that four-fermion interactions are nonrenormalizable. Could it be that the nonrenormalizable divergences they generate cancel the divergences associated with the usual perturbative treatment of gravity? The equivalence exhibited in this paper would seem to imply that this is indeed the case; however, it is far from trivial to prove it

explicitly. If this is true we can speculate further, and note that such a cancellation of divergences has a distinct flavour of supersymmetry about it. Could it be that the fermionic degrees of freedom we are postulating result from an underlying (super)-symmetry principle, capable of replacing diffeomorphism or refoliation invariance? An answer in the affirmative would explain many mysteries pertaining to HL theory, and why it works so well. This intriguing possibility, however, remains a conjecture.

Finally, we should emphasize that in our formulation time diffeomorphism invariance (refoliation invariance) is not explicitly broken. It is only spontaneously broken, as much as our Universe and the undeniable existence of a cosmological frame are bound to minimally break it. This will necessarily soften the more unwanted implications of HL theory. We conjecture, in particular, that a closer analysis of our model should reveal an absence of the

scalar graviton mode plaguing the theory. In addition this seems to be possible without the need to introduce extra symmetries, such as in Ref. [2]. We defer to a future paper an extensive analysis of this issue.

To summarize, Hořava's theory can be seen as a specific case of the covariant first-order gravity theory (Einstein-Cartan-Kibble-Holst). When the covariant theory is rewritten in Ashtekar variables, the imposition of the York time yields the Hořava theory with the Cotton tensor, in the presence of a fermion aether which breaks time refoliation invariance.

ACKNOWLEDGMENTS

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