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Complete set of representations for dissipative chaotic three-dimensional dynamical systems

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Embeddings are diffeomorphisms between some dynamical phase space and a reconstructed image. Different embeddings may or may not be equivalent under isotopy. We regard embeddings as representations of the dynamical phase space. We determine the topological labels required to distinguish inequivalent representations of three-dimensional dissipative dynamical systems when the embeddings are into \( \mathbb{R}^4 \), \( k=3,4,5,\ldots \). Three representation labels are required for embeddings into \( \mathbb{R}^3 \), and only one is required in \( \mathbb{R}^4 \). In \( \mathbb{R}^5 \) there is a single “universal” representation.

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I. INTRODUCTION

Dynamical systems are generally studied through their embeddings into the Euclidean spaces \( \mathbb{R}^k \). This is particularly true for data generated by a chaotic dynamical system. Each embedding is a diffeomorphism relating the original system and its attractor with an image. The image can be regarded as a representation of the original system and its attractor. Since it is obtained through a diffeomorphism, the representation is faithful.

As in the theory of groups, it is useful to study the spectrum of representations available to an \( n \)-dimensional dissipative dynamical system when embedded into \( \mathbb{R}^k \), \( k \geq n \). With such information it is possible to address the following important question that has never been adequately addressed: when we study an embedding of a dynamical system, how much of what we learn depends on the embedding and how much depends on the system alone, i.e., is embedding independent?

This question has already been addressed, and representation theory already worked out, for a special but large and important class of dynamical systems, those of genus-one type in \( \mathbb{R}^2 \) [1]. In the present paper, we extend this program of representation theory to a larger class: three-dimensional dynamical systems of arbitrary genus. More specifically, we study embeddings of dissipative three-dimensional dynamical systems (flows) that generate strange attractors and satisfy the conditions of the Birman-Williams theorem [2,3]. Our results are summarized in Table II.

We are careful here to distinguish two uses of the term embedding in dynamical system theory. The first usage means a diffeomorphism of a manifold onto a subset of another [4]. It is a one-to-one smooth mapping with a smooth inverse. The second usage means a one-to-one mapping of the attractor (usually not a manifold) of a dynamical system onto its image. This is the usual sense in the context of reconstructing dynamical systems [5,6]. An embedding of the first type induces an embedding of the second. In this paper, embedding will always be used in the first sense.

In Sec. II we briefly review the spirit of the representation theory of groups and compare it to the spirit of the representation theory of dynamical systems. The similarities and differences are pointed out. In Sec. III we summarize the results obtained in a previous study of the genus-one case. Many of the results for the genus-\( g \) case are simple extensions of the previous results. In Sec. IV we present our results. They are presented by comparison with the genus-one results and the motivations underlying these extensions are described. Detailed proofs are not given in this paper: they are available elsewhere [7]. We summarize our results in Sec. V.

II. IDEA OF REPRESENTATION THEORY

The theory of groups is closely associated with the theory of group representations [8]. A representation of a group is a mapping of each group operation into a matrix that preserves the group operations. The mapping can be either an isomorphism (1:1 or faithful) or a homomorphism (many:1 or unfaithful). Equivalence of representations is by similarity transformation. For computational purposes, it is sometimes simpler to carry out calculations in a faithful matrix representation rather than in the group itself.

A representation theory for dynamical systems is under construction [1,7]. At present it has been implemented only for three-dimensional dissipative dynamical systems that satisfy the conditions of the Birman-Williams theorem. It is similar in spirit but different in details from the more familiar representation theory for groups. The basic idea is that a dynamical system can be mapped onto another dynamical system through a smooth mapping. The image system is a representation of the original dynamical system. The mapping can be a diffeomorphism (1:1 or faithful) or not. We consider only faithful representations. Equivalence of representations is by isotopy—we consider two representations of a dynamical system to be equivalent if they are isotopic. Roughly speaking, two embeddings are isotopic if one can be smoothly deformed into the other through a continuous sequence of embeddings (no tearing or gluing allowed).

As for the representation theory of groups, several questions pose themselves. Given a dynamical system, what is the spectrum of inequivalent representations? What representation labels are required to distinguish among inequivalent representations? Other natural questions have no analog in the theory of group representations: as the embedding dimension increases, some representations that are inequa-
lent in the lower-dimensional space become equivalent in the higher-dimensional space. Which ones?

Questions of this type are important when considering the reconstruction of dynamical systems through embedding methods. Different embeddings may provide inequivalent representations. As the dimension of the embedding space increases, representations that were inequivalent in a lower-dimensional embedding become equivalent in higher dimensions. There is a limit to this. Takens [5] showed that an $n$-dimensional dynamical system can always be embedded in $\mathbb{R}^{2n+1}$ using a differential or time delay mapping based on a generic observable, and Wu [9] showed that all embeddings (representations) of an $n$-dimensional manifold [10] ($n \geq 2$) are isotopic (equivalent) in $\mathbb{R}^{2n+1}$. As a result, an $n$-dimensional dynamical system can have many inequivalent representation in $n$ dimensions, and as the dimension $k$ increases, the number of inequivalent embeddings decreases. Finally, for $k \geq 2n+1$ there is only one “universal” representation or embedding. A single universal representation may exist for a smaller value of the embedding dimension $k$, as in the cases discussed below.

### III. REVIEW OF GENUS ONE

The details of the representation theory of three-dimensional dissipative dynamical systems of genus one have already been worked out. We review them briefly here as the extension of our results to three-dimensional dynamical systems of higher genus is closely related to the genus-one results. Such dynamical systems are flows inside a torus $T=D^2 \times S^1$. The results are obtained by a combination of topological arguments combined with dynamical properties that “dress” the torus with a flow.

Three labels are required to distinguish inequivalent representations in $\mathbb{R}^3$. These are oriented knot type $K$, parity $Z_2$, and global torsion $Z$. The knot type of the representation is the knot type of the centerline or core curve of the torus. This core knot may be obtained by simultaneously shrinking each disk $D^2$ in $T=D^2 \times S^1$ to its center, yielding the circle $\{0\} \times S^1$. This circle inherits an orientation determined by the direction of the original flow. This oriented core may be thought of as describing the fundamental flow direction of the dynamical system. This fundamental flow direction may be mapped onto any oriented knot in $\mathbb{R}^3$. We denote by $K$ the set of all oriented knots. Different knots determine different (inequivalent) representations.

The parity of the representation is its handedness or orientation. The orientation reversing diffeomorphism $(x,y,z) \mapsto (x,y,-z)$ of $\mathbb{R}^3$ changes the handedness of the representation. There are exactly two orientations ($Z_2=\{\pm 1\}$). Different orientations determine different representations.

Global torsion is more subtle [1,11]. A genus-one system has a global Poincaré section consisting of a disk $D$ transverse to the flow. Imagine cutting open $T$ along this disk, rotating one side of the cut $q$ turns, then reconnecting. If the number of rotations is an integer, the flow will always match up continuously afterward; however, a little care must be taken to ensure smoothness. The integer $q \in Z$ is the global torsion. Different global torsions determine different representations.

### IV. EXTENSION TO HIGHER GENUS

In this section we present the analogous representation theory for three-dimensional dynamical systems with genus greater than one [13]. We justify these results by their similarity to the results in the genus-one case. Proofs of these results are available elsewhere [7].

Genus-$g$ attractors (attractors with $g$ “holes”) and the branched manifolds that describe them can be embedded in a three-dimensional manifold by a process of “inflation” [13]. Perform the inflation by surrounding each point in the branched manifold with a small $\varepsilon$ ball and taking the union of all balls. For sufficiently small $\varepsilon$ the resulting object is a manifold, specifically a genus-$g$ handlebody. This manifold serves as a proxy for the original unseen phase space. The $g$ holes correspond to the splitting regions of the original flow [3,13], often indicating fixed points [e.g., Lorenz (Fig. 1)].

The boundary of a genus-$g$ handlebody is a genus-$g$ surface called a bounding torus [14]. A genus-one system is bound by the torus $(S^1 \times S^1)$, and the genus-one handlebody is the solid torus $T$. We now study how the phase spaces of
these genus-$g$ attractors can be embedded into $R^3, R^4, \ldots$

As a nontrivial example, the Lorenz system (with typical
control parameter values $[[15]]$) has a genus three attractor that
lives inside a genus three handlebody, which is shown in Fig.
1. The caricature of this flow, its Birman-Williams projection
or template $[[2,3,16]]$, is also shown.

It is possible to construct handlebodies as the union of
basic building blocks called trinions $[[17]]$, which are fundamental
not only to the topology but also to the dynamics. These trinions are $Y$ junctions, and they come in two types: splitting and joining. These two types correspond to the two
fundamental units of templates: splitting and joining charts (see Fig. 2). Trinions are obtained by taking the inflation of
either type of chart in the branched manifold. Each trinion
has three ports, which are disks $D^2$ to which the flow is
always transverse. Splitting trinions are regions where a flow
is split into two separate streams; they have one input port
and two output ports. Joining trinions are regions where two
separate streams are mixed together; they have two inputs
and one output. Just as templates are built “Lego©” style by
connecting splitting and joining charts $[[18]]$, handlebodies are
built by connecting trinions. This construction is subject to
the following two constraints: (i) output ports of splitting trinions
flow to input ports of joining trinions; (ii) output
ports of joining trinions flow to input ports of splitting trinions.
A genus-$g$ handlebody is created by gluing together
$2(g-1)$ trinions: $g-1$ splitting and $g-1$ joining. It is this
decomposition of the genus three handlebody into four
trinions that is shown in Fig. 1.

We may now enumerate the representations of genus-$g$
systems in $R^3$. The first representation label is oriented knot
type $K_g$, which is obtained as follows. A genus-$g$ handlebody
has a core just as in the genus-one case $[[7]]$. To obtain the
core, shrink each of the $2(g-1)$ trinions onto a three legged graph or dreibein as illustrated in Fig. 3. Each leg carries a
flow direction determined by the flow directions through the
ports of the trinion. The dreibein for a splitting (joining)
trinion has 1 (2) inflowing leg and 2 (1) outflowing legs. The
core of the handlebody is the union of the dreibein along
common edges. The result is a directed graph of genus $g$

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common edges. The result is a directed graph of genus $g$

with $2(g-1)$ trivalent vertices and $3(g-1)$ edges. This directed graph represents the fundamental flow directions of the dynamical system, just as the core circle did for genus-one flows. The collection of all knot types $K_g$ is the set of all embeddings of these genus-$g$ graphs into $R^3$.

The second label is again parity. A handlebody has an orientation and the mapping $(x,y,z)\mapsto(x,y,-z)$ reverses it. There are exactly two orientations, $Z_2=\{\pm1\}$, just as in the genus-one case $[[7]]$.

The last representation label is the analog of global torsion. A genus-$g$ handlebody is constructed by gluing $g-1$
splitting and $g-1$ joining trinions together. Altogether, there are $3(g-1)$ such gluings [corresponding to the $3(g-1)$ graph edges above]. The flow is always transverse to the port disks $D_i$ where trinions are glued together. The union of the output disks of the joining trinions may be taken as a global Poincaré section for the flow $[[13]]$. The handlebody may be cut along any of these disks and one side rotated $q_i \in Z$ turns before being reconnected. The result is a spectrum of $3(g-1)$ local torsions $(q_1,\ldots,q_{3(g-1)}) \in Z^{3(g-1)}$.

In three dimensions, in direct analogy with the genus-one case, there is a triple of representation labels $(K_g,Z_2,Z^{3(g-1)})$; the oriented knot type, an orientation, and a spectrum of local torsions $[[7]]$. We point out that the problem of distinguishing two knotted circles in $R^3$ is difficult and still has no general solution. The corresponding problem for higher genus knotted graphs is correspondingly more difficult. Nevertheless, in simple cases it may be reasonable to distinguish embedded graphs “by inspection.”

In $R^4$ many of these distinct representations become
equivalent as obstructions to isotopy are lifted $[[1]]$. Since
graphs are essentially one-dimensional objects, all of their
embeddings in $R^4$ are isotopic, just as all embedded closed
curves are isotopic. Knotted graphs become unknotted just as
toroidal circles do $[[7]]$. Oriented knot type is no longer a
representation label. Parity also ceases to distinguish embeddings.

Once again, local torsion is more subtle $[[7]]$. We anticipate
that, in analogy with the genus-one case, the local torsions at
each of the $3(g-1)$ ports fall into two classes: $q_i$ even and $q_i$ odd. Thus, the integer $Z$ that characterized the torsion at each port is reduced to $Z_2=\{0,1\}$. But this is not all. On any trinion, a single twist on any port can be translated into a pair of twists—one in each of the other two ports—at the expense of introducing a writhe or twisting of the legs near those ports. However, this writhing is easily pulled apart in $R^4$ (see Fig. 4). This means that a single twist on any one port is fully
necessary to distinguish among inequivalent representations since the topological nature of suitable invariants in higher-dimensional dynamical systems, the representation label, a spectrum of 2\(^{g-1}\) local torsions; one in \(\mathbb{R}\)\(^6\), \(\mathbb{R}\)\(^7\), and none in \(\mathbb{R}\)\(^5\). However, it may also exist in lower dimensions: five for three-dimensional systems.

The representation theory for three-dimensional dynamical systems has been developed in two steps. In an earlier contribution we studied the representation theory of dynamical systems of “genus-one” type. This class includes nonautonomous dynamical systems such as periodically driven two-dimensional nonlinear oscillators and autonomous three-dimensional dynamical systems such as the Rössler attractor. All these systems have the solid torus \(T\) as their natural phase space, and their chaotic dynamics is generated by “stretching and folding.” For these systems there are three representation labels in \(\mathbb{R}\)\(^3\): knot type, parity, and global torsion; one in \(\mathbb{R}\)\(^4\): global torsion (mod 2) and none in \(\mathbb{R}\)\(^5\). The universal representation exists in \(\mathbb{R}\)\(^6\), \(k \geq 5\). This information is summarized in Table II.

The present work completes this representation theory for all three-dimensional “genus-g dynamical systems” and their attractors. This class includes the Lorenz attractor and other systems whose chaotic dynamics are generated by “tearing and squeezing.” These systems have genus-g handlebodies as phase space. For these systems there are three representation labels in \(\mathbb{R}\)\(^3\): knot type, parity, and a spectrum of local torsions; one in \(\mathbb{R}\)\(^4\): a reduced spectrum of local torsions (mod 2) and none in \(\mathbb{R}\)\(^5\). The universal representation exists in \(\mathbb{R}\)\(^6\), \(k \geq 5\). This information is summarized in Table II.

These results allow us to conclude that some of the information obtained by analyzing a three-dimensional embedding of a three-dimensional dynamical system depends on the embedding (knot type, parity, and global or local torsion) and some are embedding independent. However, since the universal embedding exists in \(\mathbb{R}\)\(^3\) (or higher), any information extracted from such an embedding must be intrinsic—it must depend on the dynamics alone and not at all on the embedding.

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![Figure 4](image-url)
[10] Technically, Wu’s theorem refers to manifolds without boundary, so it does not immediately apply to the handlebodies considered here. However, our results are independent of Wu’s theorem, and we still expect a universal embedding for arbitrary manifolds with boundary in some dimension.