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Five-Dimensional Space-Times

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Behavior of homogeneous five-dimensional space-times

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The set of all spatially homogeneous (4 + 1)-dimensional cosmologies will be explored. We will utilize a classification of these space-times based on the actions of isometry groups on these manifolds. Several exact solutions of the Einstein equations will be derived. The possibilities of dimensional reduction and isotropization will be examined. Five-dimensional generalizations of the mixmaster universe will be developed. Finally, the Einstein equations in 4 + 1 dimensions will be viewed as a Hamiltonian system. The absence of chaotic solutions to these equations will be shown.

I. INTRODUCTION

Higher-dimensional models have come into increasing popularity as unified field theories. In these theories, the scales of the extra dimensions are considered to be presently very small. Chodos and Detweiler[1] have proposed a model in which one or more of these extra dimensions contract. They considered Einstein's field equations in five dimensions, which for the vacuum cases are

$$R_{\mu\nu}=0\ , \tag{1}$$

where $R_{\mu\nu}$ is the five-dimensional Ricci tensor.

They considered a five-dimensional Kasner-type solution to these equations:

$$ds^2=-dt^2+\sum_{i=1}^{4}(t/t_0)^{2p_i}(dx^i)^2\ , \tag{2}$$

where $\sum p_i=\sum p_i{}^2=1$. In order to guarantee the appearance of isotropy they took $p_1=p_2=p_3=-p_4=\frac{1}{2}$. This model contracts monotonically in one spatial dimension, and expands monotonically and equally in the other three.

An interesting question in cosmology is why the Universe has three spatial dimensions and not some other number. An explanation, based on the anthropic principle, has been proposed. Barrow[2] has pointed out the role played by the dimensions of space-time in determining the form of various physical laws and constants. Perhaps intelligent life can only emerge in a four-dimensional space-time?

The model of Chodos and Detweiler is not the most general (4 + 1)-dimensional model in that it is isotropic in its three expanding spatial dimensions for all time. This assumption about the state of the early Universe may not be necessary.

The question of whether or not the Universe has always been isotropic and homogeneous has been a central issue in cosmology for quite some time. The fact that local inhomogeneities, such as galaxies, exist suggests that the isotropy and homogeneity of the Universe may have been a later development. One therefore would like to establish bounds on the anisotropy of the early Universe given the experimental limits of our current measurements.

Collins and Hawking[3] in their classic paper considered the set of possible early states of the Universe which could lead to the present-day isotropy and homogeneity. The set of initial data which they examined was the set of all spatially homogeneous and three-dimensional cosmologies. These data include a three-index tensor $C^a{}_{bc}$, which defines the action of an isometry group on the initial surface. This tensor can be classified into one of ten types, called Bianchi types. Each of these ten equivalence classes of tensors forms a submanifold in the space of all three-index tensors. Only four of these types, corresponding to submanifolds of largest dimension, form a set of nonzero measure in the space of all initial data. Collins and Hawking showed that the set of all initial data leading to isotropy at later times is of measure zero. Thus, an isotropic universe is dynamically unstable under the action of all homogeneous perturbations.

In order to explain the improbability that the Universe is isotropic today, given this set of possible initial conditions, Collins and Hawking advocated applying the anthropic principle. One considers the set of all possible universes. Only in universes which have very nearly the escape velocity can galaxies be formed. Collins and Hawking found that these universes approach isotropy at large times. Since the existence of galaxies is probably necessary for the development of intelligent life, our existence in the galaxy implies that the Universe is isotropic.

In order to fully generalize this, one might extend our set of initial data to include universes of higher dimension. Let us now consider the set of all possible universes of any number of dimensions. A subset of this collection possesses the following properties.

(1) Only three spatial dimensions are observable, after a sufficient amount of time.

(2) After a sufficient amount of time, any anisotropy that may exist is within today's experimental limits.

An interesting and well-defined program is the determination of the size of this subset. We first place some limits on the set of initial data. We limit this set to homogeneous cosmologies. Also, for the purposes of this study, we shall only consider five-dimensional space-times. We do this for several reasons. First of all, the original higher-dimensional unified field theories had 4 + 1 dimensions. These theories reduce to the Einstein-

Maxwell equations. Second, the structure constants and Killing vectors for the homogeneous five-dimensional space-times have been cataloged. Third, some of these cosmologies have already been studied. Finally, some of our results have implications for higher-dimensional models.

The real four-dimensional Lie algebras have been classified by different authors. We shall use the classification given by Fee.[4] In this classification, there are 15 distinct real four-dimensional Lie algebras, called $G0$—$G14$ (see Table I). The set of structure constants of each of these groups form submanifolds in the space of all three-index tensors.

The maximum dimensionality of these submanifolds is 9. Nine of the Fee algebras have the distinction of corresponding to submanifolds of this dimension. The remaining ones, $G0$, $G1$, $G2$, $G3$, $G4$, and $G5$, are not general in the sense that they form a set of zero measure in the space of three-index tensors. $G0$ corresponds to a submanifold to 0 dimension, for instance.

We shall examine the solutions of the Einstein equations for some of these types and determine whether or not, in these models, three dimensions expand monotonically and at equal rates while one contracts to unobservability, given a long enough amount of time.

Also we shall examine the dynamical properties of the Einstein equations as a Hamiltonian system. To do this, we extend the powerful technique of Arnowitt, Deser, and Misner to five dimensions. We compare the dynamical properties of (3 + 1)-dimensional and (4 + 1)-dimensional solutions of the Einstein equations. Are there any properties of the solutions of these equations which are exhibited in the (3 + 1)-dimensional case, but not in higher dimensions?

II. SOME SIMPLE FIVE-DIMENSIONAL MODELS

We now will find exact vacuum solutions for some of the five-dimensional homogeneous cosmologies. We utilize Fee's classification of the four-dimensional Lie groups. The right- and light-invariant vector fields and forms are given in Fee's work. We write the metric, in each case, in the Cartan basis of left-invariant forms:

$$ds^2=-dt^2+g_{ij}\omega^i\omega^j, \quad g_{ij}=g_{ij}(t) . \tag{3}$$

The one-forms ω^i obey

$$d\omega^i=\tfrac{1}{2}C^i{}_{st}\omega^s\wedge\omega^t , \tag{4}$$

where the $C^i{}_{st}$ are the structure constants of the appropriate group. In each case we assume the metric to be diagonal:

$$g_{ij}=\mathrm{diag}(a^2,b^2,c^2,d^2) . \tag{5}$$

We can now write out the Einstein equations:

$$-R^0_0=\frac{\ddot a}{a}+\frac{\ddot b}{b}+\frac{\ddot c}{c}+\frac{\ddot d}{d}=0 , \tag{6}$$

$$-R^1_1=\frac{(\dot abcd)^{\cdot}}{(abcd)}+S^1{}_1=0 , \tag{7}$$

$$-R^2_2=\frac{(a\dot bcd)^{\cdot}}{(abcd)}+S^2{}_2=0 , \tag{8}$$

$$-R^3_3=\frac{(ab\dot cd)^{\cdot}}{(abcd)}+S^3{}_3=0 , \tag{9}$$

$$-R^4_4=\frac{(abc\dot d)^{\cdot}}{(abcd)}+S^4{}_4=0 , \tag{10}$$

$$R^0_n=\left[\frac{\dot x_n}{x_n}-\frac{\dot x_m}{x_m}\right]C^m_{mn}=0 , \tag{11}$$

where a dot denotes d/dt. The x_n ($n=1,2,3,4$) denote the appropriate scale factor: a, b, c, or d. The $S^n{}_n$ are functions of a, b, c, d and the structure constants. A convenient way of writing the Einstein equations is to replace the variable t with τ in accordance with

$$dt=\Lambda d\tau, \quad \Lambda=abcd . \tag{12}$$

Now let us introduce instead of a, b, c, and d, their logarithms α, β, γ, and δ, respectively. Then the Einstein equations become

TABLE I. Four-dimensional Lie algebras.

$G0$	All commutators are zero			
$G1$	$[x,z]=w$			
$G2$	$[x,z]=w$	$[y,z]=w$		
$G3$	$[x,z]=w$	$[y,z]=y$		
$G4$	$[w,y]=w$	$[x,z]=x$		
$G5$	$[w,y]=-x$	$[w,z]=w$	$[x,y]=w$	$[x,z]=w$
$G6$	$[w,z]=w$	$[x,z]=w+x$	$[y,z]=x+y$	
$G7$	$[w,z]=w$	$[x,z]=w+x$	$[y,z]=py$	
$G8$	$[w,z]=w$	$[x,z]=px$	$[y,z]=qy$	
$G9$	$[w,z]=pw+x$	$[x,z]=-w+px$	$[y,z]=qy$	
$G10$	$[w,z]=2w$	$[x,y]=w$	$[x,z]=x$	$[y,z]=x+y$
$G11$	$[w,z]=(1+p)w$	$[x,y]=w$	$[x,z]=x$	$[y,z]=py$
$G12$	$[w,z]=2pw$	$[x,y]=w$	$[x,z]=px+y$	$[y,z]=-x+py$
$G13$	$[w,x]=y$	$[w,y]=-x$	$[x,y]=w$	
$G14$	$[w,x]=y$	$[w,y]=-x$	$[x,y]=-w$	

$$\begin{aligned}
\alpha_{\tau\tau}&=-\Lambda^2 S^1{}_1 ,\\
\beta_{\tau\tau}&=-\Lambda^2 S^2{}_2 ,\\
\gamma_{\tau\tau}&=-\Lambda^2 S^3{}_3 ,\\
\delta_{\tau\tau}&=-\Lambda^2 S^4{}_4 ;
\end{aligned} \tag{13}$$

$$\alpha_{\tau\tau}+\beta_{\tau\tau}+\gamma_{\tau\tau}+\delta_{\tau\tau}=2\alpha_\tau\beta_\tau+2\alpha_\tau\gamma_\tau+2\alpha_\tau\delta_\tau+2\beta_\tau\gamma_\tau+2\beta_\tau\delta_\tau+2\gamma_\tau\delta_\tau ; \tag{14}$$

and

$$R^0_n=0 \quad (n=1,2,3,4) . \tag{15}$$

In the case of the space-time corresponding to the $G0$ Fee group, we have the simple generalized Kasner model. The Einstein equations are

$$\begin{aligned}
\alpha_{\tau\tau}&=0 ,\\
\beta_{\tau\tau}&=0 ,\\
\gamma_{\tau\tau}&=0 ,\\
\delta_{\tau\tau}&=0 ,
\end{aligned} \tag{16}$$

and

$$\alpha_{\tau\tau}+\beta_{\tau\tau}+\gamma_{\tau\tau}+\delta_{\tau\tau}=2\alpha_\tau\beta_\tau+2\alpha_\tau\gamma_\tau+2\alpha_\tau\delta_\tau+2\beta_\tau\gamma_\tau+2\beta_\tau\delta_\tau+2\gamma_\tau\delta_\tau .$$

The solution of this is the Chodos and Detweiler model mentioned in the last section. Note that the submanifold corresponding to the group $G0$ has zero dimensions in the space of all three-index tensors. Also, only one set of parameters for this model exhibits the long-term behavior of isotropy in the three observable spatial scale factors. Therefore this model has zero measure in the space of all possible initial conditions.

For the $G1$ space-time the Einstein equations are

$$\begin{aligned}
2\alpha_{\tau\tau}&=-\exp(4\alpha+2\delta) ,\\
2\beta_{\tau\tau}&=\exp(4\alpha+2\delta) ,\\
2\gamma_{\tau\tau}&=\exp(4\alpha+2\delta) ,\\
\delta_{\tau\tau}&=0 ,
\end{aligned} \tag{17}$$

and

$$\alpha_{\tau\tau}+\beta_{\tau\tau}+\gamma_{\tau\tau}+\delta_{\tau\tau}=2\alpha_\tau\beta_\tau+2\alpha_\tau\gamma_\tau+2\alpha_\tau\delta_\tau+2\beta_\tau\gamma_\tau+2\beta_\tau\delta_\tau+2\gamma_\tau\delta_\tau .$$

The solution to this is

$$\begin{aligned}
a^2&=\exp(2\alpha)=A\omega\,\mathrm{sech}(\omega\tau)\exp(-k_1\tau) ,\\
b^2&=\exp(2\beta)=B\cosh(\omega\tau)\exp(k_2\tau) ,\\
c^2&=\exp(2\gamma)=C\cosh(\omega\tau)\exp(k_3\tau) ,\\
d^2&=(1/A^2)\exp(2k_1\tau) ,
\end{aligned} \tag{18}$$

where

$$\omega^2=-2k_1{}^2+k_1k_2+k_1k_3+k_2k_3 ,$$

and k_2, k_3, A, B, and C are all constants.

Note that the asymptotic behavior of this solution as $\tau\to\infty$ and $\tau\to-\infty$ is that of a generalized Kasner model. Therefore, only one set of initial Kasner parameters leads to long-term anisotropy. This model also has zero measure in the set of all possible initial data.

The Einstein equations for the $G2$ space-time are

$$\begin{aligned}
2\alpha_{\tau\tau}&=-\exp(4\alpha+2\delta)+\exp(4\delta+2\beta) ,\\
2\beta_{\tau\tau}&=\exp(4\alpha+2\delta) ,\\
2\gamma_{\tau\tau}&=\exp(4\alpha+2\delta)+\exp(4\delta+2\beta) ,\\
2\delta_{\tau\tau}&=-\exp(4\alpha+2\delta) ,
\end{aligned} \tag{19}$$

and

$$\alpha_{\tau\tau}+\beta_{\tau\tau}+\gamma_{\tau\tau}+\delta_{\tau\tau}=2\alpha_\tau\beta_\tau+2\alpha_\tau\gamma_\tau+2\alpha_\tau\delta_\tau+2\beta_\tau\gamma_\tau+2\beta_\tau\delta_\tau+2\gamma_\tau\delta_\tau .$$

From these equations one can show that

$$a=\exp(\alpha)=k_1\exp(k_2\tau-\beta-\delta)$$

and (20)

$$c=\exp(\gamma)=k_3\exp(k_4\tau+\beta-\delta) .$$

Thus the Einstein equations (19) reduce to

$$\begin{aligned}
&2\beta_{\tau\tau}=(k_1)^4\exp(-4\beta-2\delta+4k_2\tau) ,\\
&2\delta_{\tau\tau}=-\exp(4\delta+2\beta) ,\\
&\beta_{\tau\tau}-\delta_{\tau\tau}+2(\beta_\tau)^2+2(\delta_\tau)^2+2\delta_\tau\beta_\tau-2k_2k_4-4k_2\beta_\tau=0 .
\end{aligned} \tag{21}$$

One particular solution to these equations is

$$\begin{aligned}
a&=\exp(\alpha)=k_1 ,\\
b&=\exp(\beta)=k_2\tau ,\\
c&=\exp(\gamma)=k_4\exp(k_3\tau)\tau^2 ,\\
d&=\exp(\delta)=k_5/\tau ,
\end{aligned} \tag{22}$$

with

$$k_1{}^4=k_5{}^2k_2{}^2,\quad k_1{}^4k_5{}^2=2 .$$

Note that this model does not expand in three spatial dimensions and thus is not a candidate for a Kaluza-Klein cosmology.

The $G3$ model is interesting in that its Einstein equations do not admit a solution with four nonzero scale factors. The Einstein equations for this model are

$$-R^0_0=\frac{\ddot a}{a}+\frac{\ddot b}{b}+\frac{\ddot c}{c}+\frac{\ddot d}{d}=0 , \tag{23}$$

$$-R^1_1=\frac{(\dot abcd)^\cdot}{(abcd)}=\frac{1}{2}\frac{d^2}{a^2c^2}=0 , \tag{24}$$

$$-R^2_2=\frac{(a\dot bcd)^\cdot}{(abcd)}-\frac{1}{c^2}=0 , \tag{25}$$

$$-R^3_3=\frac{(ab\dot cd)^\cdot}{(abcd)}-\frac{1}{2}\frac{d^2}{a^2c^2}-\frac{1}{c^2}=0 , \tag{26}$$

$$-R^4_4=\frac{(abc\dot{d})^{\cdot}}{(abcd)}+\frac{1}{2}\frac{d^2}{a^2c^2}=0\ , \tag{27}$$

$$R^0_3=\left[\frac{\dot{c}}{c}-\frac{\dot{b}}{b}\right]=0\ . \tag{28}$$

From Eq. (28), $c=kb$, where k is a constant. Then by subtracting Eq. (25) from Eq. (26), one finds that $d=0$. Therefore the general solution of these equations is the four-dimensional Kasner solution:

$$\begin{aligned} &a=At^p\ ,\\ &b=c=Bt^q\ ,\\ &d=0\ , \end{aligned} \tag{29}$$

where

$$p=-\tfrac{1}{3}\ \text{ and } q=-\tfrac{2}{3}\ .$$

The Einstein equations for the $G4$ universe include the identities

$$\alpha_\tau=\gamma_\tau,\ \ \beta_\tau=\gamma_\tau\ . \tag{30}$$

Therefore, for this model,

$$c=k_1a,\ \ d=k_2b\ ,$$

where k_1 and k_2 are constants. This is clearly not a valid Kaluza-Klein cosmology in that either two of the scale factors expand, while two contract, or all four expand.

Solutions have been also found for the $G7$, $G8$, and $G11$ five-dimensional space-times as classified by Fee.[5] Some of these solutions have the behavior that three of the spatial dimensions expand monotonically, while one contracts monotonically. However, in each case, the dependence of each of the scale factors on time is vastly different from that of the other two. In each case, the Universe models become increasingly more anisotropic, sometimes at an exponential rate.

We can draw some interesting conclusions from this study. First of all, unlike the (3 + 1)-dimensional case, not all of the (4 + 1)-dimensional vacuum Einstein equations for the set of all equivalence classes of structure constants admit solutions. Of this set of solutions, may exhibit the undesirable property of contraction in more than one scale factor, or expansion in more than three. Finally, of this set, the solutions which approach isotropy are probably of measure zero.

III. THE FIVE-DIMENSIONAL MIXMASTER MODEL

In recent years, several authors have applied dynamical systems theory to the study of the solutions of the Einstein equations. One important concept in this theory is the idea of chaos. Chaos may be viewed as a situation in which a dynamical system, without the introduction of stochastic forcing or other random elements, exhibits behavior which is, for all intents and purposes, unpredictable. One would have to know the initial data with absolute certainty in such systems, since solutions with neighboring sets of initial data diverge.

Among the set of four-dimensional anisotropic space-times, the property of chaos distinguishes two Bianchi-type models. Bianchi types VIII and IX are the only homogeneous vacuum models which possess a nonzero metric entropy, an indicator of chaos.[6]

The way that this chaos manifests itself is through a series of oscillations as a universe of this type collapses towards a singular state. In the evolution of the Bianchi-type-IX model, two of the universe's scale factors are oscillating at any given time, while the third is experiencing exponential decay. After a transition period, one of the oscillating scale factors switches with the decaying scale factor, and a new "era" begins. The length of each era is dependent on the length of its predecessor. However, a small change in the length of one era leads to such a large change in the length of its successor, that the appearance is that of a random-number generator.

It is interesting to examine whether or not higher-dimensional cosmologies might possess chaotic solutions. Several authors have shown that none of the higher-dimensional extensions of the mixmaster model (Bianchi type IX) are chaotic.[7] Mixmaster models are universes which display a pattern of oscillation similar to that of the Bianchi type-IX model. However, if the number of oscillations is finite, then the Universe is demonstrably not chaotic.

Let us look at the three five-dimensional homogeneous cosmologies which display behavior which could be characterized as "mixmaster." The five-dimensional model analogous to Bianchi type IX is the Fee model $G13$. The structure constants for this model are the same as that of Bianchi type IX.

The Einstein equations for this model can be written

$$\begin{aligned} &2\alpha_{\tau\tau}=[(b^2-c^2)^2-a^4]d^2\ ,\\ &2\beta_{\tau\tau}=[(c^2-a^2)^2-b^4]d^2\ ,\\ &2\gamma_{\tau\tau}=[(a^2-b^2)^2-c^4]d^2\ ,\\ &\delta_{\tau\tau}=0\ ; \end{aligned} \tag{31}$$

$$\begin{aligned} \alpha_{\tau\tau}+\beta_{\tau\tau}+\delta_{\tau\tau}= {}& 2\alpha_\tau\beta_\tau+2\alpha_\tau\gamma_\tau+2\alpha_\tau\delta_\tau\\ &+2\beta_\tau\gamma_\tau+2\beta_\tau\delta_\tau+2\gamma_\tau\delta_\tau\ . \end{aligned} \tag{32}$$

When we can neglect the right-hand side of Eqs. (31), the solution of these equations is a five-dimensional Kasner space-time. This is the situation in the limit $\tau\to\infty$ ($t\to\infty$). Since our system is considered to be evolving backward in time, our initial conditions can be written in terms of the Kasner parameters:

$$\begin{aligned} &\alpha=\Lambda p_1\tau+\text{const}\ ,\\ &\beta=\Lambda p_2\tau+\text{const}\ ,\\ &\gamma=\Lambda p_3\tau+\text{const}\ ,\\ &\delta=\Lambda p_4\tau+\text{const}\ , \end{aligned} \tag{33}$$

where

$$\sum p_i=\sum {p_i}^2=1,\ \ i=1,4\ .$$

The Universe remains in the Kasner regime only when the terms on the right-hand side of Eqs. (31) are small. However, as τ decreases, one or more of the terms may increase. We then may no longer neglect the right-hand side of the equations. Assume that a is this term, and that it is larger than the other terms. Then we can write the equations (31) as

$$\alpha_{\tau\tau}=-\tfrac{1}{2}\exp(4\alpha+2\delta)\ ,$$
$$\beta_{\tau\tau}=\gamma_{\tau\tau}=\tfrac{1}{2}\exp(4\alpha+2\delta)\ , \tag{34}$$
$$\delta_{\tau\tau}=0\ ,$$

where

$$a=\exp(\alpha),\quad b=\exp(\beta),\quad \text{etc.}$$

The solution of these equations (34), satisfying the initial conditions (33) in the limit $\tau\to\infty$, is

$$a^2=\frac{A(2p_1+p_4)\Lambda}{\cosh[(2p_1+p_4)\Lambda\tau]}\exp(-\tfrac{1}{2}\Lambda p_4\tau)\ ,$$
$$b^2=B\exp[(2p_2-2p_1-p_4)\Lambda\tau]\cosh[(2p_1+p_4)\Lambda\tau]\ , \tag{35}$$
$$c^2=C\exp[(2p_3-2p_1-p_4)\Lambda\tau]\cosh[(2p_1+p_4)\Lambda\tau]\ ,$$
$$d^2=A^{-2}\exp(p_4\Lambda\tau)\ ,$$

where A, B, and C are constants.

The asymptotic expressions for these functions and the function $t(\tau)$ as $\tau\to-\infty$ are

$$a\sim\exp[-(p_1+p_4)\Lambda\tau]\ ,$$
$$b\sim\exp[(p_2+2p_1+p_4)\Lambda\tau]\ , \tag{36}$$
$$c\sim\exp[(p_3+2p_1+p_4)\Lambda\tau]\ ,$$
$$d\sim\exp(p_4\Lambda\tau)\ .$$

We can then express a, b, c, and d in terms of new Kasner parameters:

$$a\sim t^{p'_1},\quad b\sim t^{p'_2},\quad c\sim t^{p'_3},\quad d\sim t^{p'_4}\ , \tag{37}$$

where

$$p'_1=-\frac{(p_1+p_4)}{(1+2p_1+p_4)},\quad p'_2=\frac{(p_2+2p_1+p_4)}{(1+2p_1+p_4)}\ ,$$
$$p'_3=\frac{(p_3+2p_1+p_4)}{(1+2p_1+p_4)},\quad p'_4=\frac{p_4}{(1+2p_1+p_4)}\ , \tag{38}$$
$$abcd=\Lambda' t,\quad \Lambda'=(1+2p_1+p_4)\Lambda\ .$$

Note that $\Sigma p'_i=\Sigma(p'_i)^2=1$, $i=1,4$.

So, the space-time undergoes a transition from one Kasner regime to another via a mechanism similar to that of the four-dimensional Bianchi type-IX model. Note, though, that this transition only takes place if one of the terms on the right-hand side of Eqs. (31) increases as $\tau\to-\infty$. As Ishihira has pointed out,[8] this is not always the case for higher-dimensional models. Thus either a^4d^2, b^4d^2, or c^4d^2 must increase for a transition to occur. Writing a, b, c, and d in terms of their initial behavior before the transition, the terms assume the Kasner regime form:

$$a^4d^2\sim t^{(1+p_1-p_2-p_3)},\quad b^4d^2\sim t^{(1+p_2-p_3-p_1)}\ ,$$
$$c^4d^2\sim t^{(1+p_3-p_1-p_2)}\ . \tag{39}$$

Therefore either

$$1+p_1-p_2-p_3>0,\quad 1+p_2-p_3-p_1>0\ , \tag{40}$$

or

$$1+p_3-p_1-p_2>0\ ,$$

for a transition to take place. Otherwise, transitions would cease, and the Universe would remain in a Kasner regime.

Let us now look at what happens to the $G13$ universe, initially in a Kasner regime parametrized by a set of Kasner parameters p_1, p_2, p_3, and p_4, as it evolves. We are only free to choose two parameters in this set, p_1 and p_2, since conditions (33) imply

$$p_4=1-p_1-p_2-p_3\ ,$$
$$p_3=\tfrac{1}{2}[(1-p_1-p_2)+(1-3p_1{}^2-3p_2{}^2-2p_1p_2+2p_1+2p_2)^{1/2}]\ . \tag{41}$$

Of course p_1 and p_2 must be such that

$$1-3p_1{}^2-3p_2{}^2-2p_1p_2+2p_1+2p_2\geq 0 \tag{42}$$

for p_3 to be real. Conditions (40) become

$$3p_1{}^2+p_2{}^2+p_1-p_2-p_1p_2>0\ ,$$
$$3p_2{}^2+p_1{}^2+p_2-p_1-p_1p_2>0\ , \tag{43}$$
$$3p_1{}^2+3p_2{}^2-5p_1-5p_2+5p_1p_2+2>0\ .$$

In order for a transition to take place, two of the initial Kasner parameters must satisfy condition (42) and one of the conditions (43). The set of allowed values for p_1 and p_2 can be graphically represented as three regions bounded by ellipses. This can be seen in Fig. 1. Outside the set depicted by (43), but within the set depicted by (42), no more transitions take place. The Universe remains in this initial Kasner state, and the behavior continues to be characterized by Eqs. (33).

If the transition does take place, then after a short interval, the Universe can be characterized by new Kasner parameters, p'_i, transformed by Eqs. (38). The same criteria can then be applied for a second transition. The new parameters must satisfy one of Eqs. (43).

The behavior of the $G14$ universe is similar to that of $G13$. The $G14$ universe has the same structure constants as Bianchi type VIII. Its Einstein equations are

$$2\alpha_{\tau\tau}=[(b^2+c^2)^2-a^4]d^2\ ,$$
$$2\beta_{\tau\tau}=[(c^2+a^2)^2-b^4]d^2\ , \tag{44}$$
$$2\gamma_{\tau\tau}=[(a^2-b^2)^2-c^4]d^2\ ,$$
$$\delta_{\tau\tau}=0\ ;$$

$$\alpha_{\tau\tau}+\beta_{\tau\tau}+\gamma_{\tau\tau}+\delta_{\tau\tau}=2\alpha_\tau\beta_\tau+2\alpha_\tau\gamma_\tau+2\alpha_\tau\delta_\tau+2\beta_\tau\gamma_\tau+2\beta_\tau\delta_\tau+2\gamma_\tau\delta_\tau\ . \tag{45}$$

Note that if one makes the approximation similar to that of $G13$, that one of the terms on the right-hand side of (44) is large compared to the other two, then one arrives at the same equations and the behavior as that of $G13$.

Another five-dimensional universe with mixmaster properties is the $G12$ universe in Fee's nomenclature. This model has no four-dimensional analogue. The Einstein equations for this universe in the diagonal case are

$$\begin{aligned}
2\alpha_{\tau\tau}&=(8P^2)a^2b^2d^2+d^4c^2+b^4d^2-a^4d^2 ,\\
2\beta_{\tau\tau}&=(8P^2)a^2b^2d^2+d^4c^2-b^4d^2+a^4d^2 ,\\
2\gamma_{\tau\tau}&=(2-6P^2)a^2b^2d^2+a^4d+b^4d^2 ,\\
2\delta_{\tau\tau}&=(16P^2)a^2b^2d^2-d^4c^2 ;
\end{aligned} \tag{46}$$

$$\alpha_{\tau\tau}+\beta_{\tau\tau}+\gamma_{\tau\tau}+\delta_{\tau\tau}=2\alpha_\tau\beta_\tau+2\alpha_\tau\gamma_\tau+2\alpha_\tau\delta_\tau+2\beta_\tau\gamma_\tau+2\beta_\tau\delta_\tau+2\gamma_\tau\delta_\tau ; \tag{47}$$

$$0=P(4\gamma_\tau-\alpha_\tau-\beta_\tau-2\delta_\tau) ; \tag{48}$$

where

$$P=C^1{}_{13}=C^2{}_{23}=\tfrac{1}{2}C^4{}_{43} .$$

Equation (48) implies that either $P=0$ or

$$4\gamma_\tau=\alpha_\tau+\beta_\tau+2\delta_\tau . \tag{49}$$

In the first case ($P=0$) the Einstein equations are

$$\begin{aligned}
2\alpha_{\tau\tau}&=d^4c^2+b^4d^2-a^4d^2 ,\\
2\beta_{\tau\tau}&=d^4c^2-b^4d^2+a^4d^2 ,\\
2\gamma_{\tau\tau}&=2a^2b^2d^2+a^4d+b^4d^2 ,\\
2\epsilon_{\tau\tau}&=-d^4c^2 ,\\
\alpha_{\tau\tau}+\beta_{\tau\tau}+\gamma_{\tau\tau}+\delta_{\tau\tau}&=2\alpha_\tau\beta_\tau+2\alpha_\tau\gamma_\tau+2\alpha_\tau\delta_\tau+2\beta_\tau\gamma_\tau+2\beta_\tau\delta_\tau+2\gamma_\tau\beta_\tau .
\end{aligned} \tag{50}$$

Now, if we make the assumption that one of the terms (let us assume that it is c) dominates two of the other terms (a,b) then the first four equations become

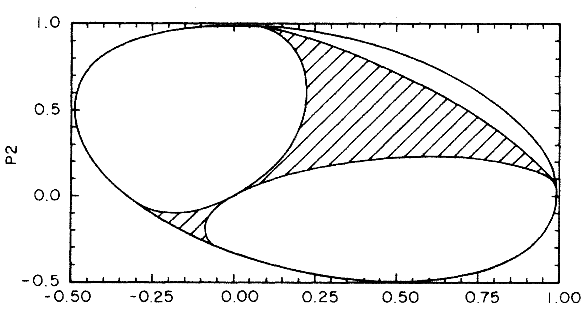

FIG. 1. Depending on the value of two of the four Kasner parameters, the $G13$ universe may or may not undergo a transition to another Kasner regime. In this graph, the large ellipse surrounds the range of possible initial Kasner values p_1 and p_2. Inside this ellipse, but outside the shaded region, transitions do occur which lead to an alteration of the Kasner parameters. However, if p_1 and p_2 are within the shaded region, the universe will undergo no future transitions.

$$\begin{aligned}
2\alpha_{\tau\tau}&=d^4c^2 ,\\
2\beta_{\tau\tau}&=d^4c^2 ,\\
2\gamma_{\tau\tau}&=0 ,\\
2\delta_{\tau\tau}&=-d^4c^2 .
\end{aligned} \tag{51}$$

These are the same equations as those of $G13$ and $G14$. If one now assumes that P is not equal to zero, then one obtains the constraint $P^2=\frac{2}{9}$ in order not to contradict one of the Einstein equations. This does not affect the asymptotic behavior of the equations in the case $c \gg a,b$. Thus, in either case, the behavior of $G12$ near a singularity is the same as $G13$ and $G14$: a finite sequence of oscillations and transitions from one Kasner state to another.

We shall now employ a different method in our analysis. This method will serve to illuminate the qualitative aspects of the behavior of higher-dimensional space-times.

IV. A HAMILTONIAN ANALYSIS OF THE FIVE-DIMENSIONAL COSMOLOGIES

The anisotropic four-dimensional cosmologies have been analyzed by utilizing the power of the Arnowitt, Deser, and Misner[9] (ADM) formulation of the Einstein variational principle. In using this method one represents the time evolution of the Universe as that of a particle (the Universe point) moving in two dimensions subject to the influence of a potential. The two axes parametrize the shape of the Universe. A different potential corresponds to each Bianchi type.

This method may also be applied to higher-dimensional cosmologies. By applying this technique, interesting information about the qualitative behavior of the universe model in question can be obtained. The shape of the potential dictates the precise nature of the evolution of the Universe.

Let us now apply this procedure in order to examine the qualitative behavior of the five-dimensional anisotropic cosmologies, as classified by Fee. We use the same form of the metric as before:

$$ds^2=-dt^2+g_{ij}\omega^i\omega^j . \tag{52}$$

We now assume that we can rewrite g_{ij}:

$$g_{ij}=\exp(2\alpha)\exp(2\beta_{ij}) , \tag{53}$$

where $\alpha(t)$ is a scalar, and $\beta_{ij}(t)$ are the elements of a traceless, diagonal matrix B.

We can reparametrize B with new variables B_0, B_+, and B_- such that

$$\begin{aligned}
\beta_{11}&=B_0+\sqrt{2}B_++\sqrt{6}B_- ,\\
\beta_{22}&=B_0+\sqrt{2}B_+-\sqrt{6}B_- ,\\
\beta_{33}&=B_0-2\sqrt{2}B_+ ,\\
\beta_{44}&=-3B_0 ,
\end{aligned} \tag{54}$$

Let us now apply the ADM formalism. The Einstein action (from which the field equations can be derived) is in the vacuum case:

$$I=(16\pi^{-1})\int {}^5R\sqrt{-g}\,d^5x\ , \tag{55}$$

where ${}^5R$ is the five-dimensional scalar curvature, and g is the determinant of the metric.

Applying the methods of ADM, this reduces to

$$I=(16\pi^{-1})\int\left[\pi^{ij}\left[\frac{\partial g_{ij}}{\partial t}\right]-NC^0-N_iC^i\right]d^5x\ . \tag{56}$$

The quantities here to be varied separately are the ij ($i,j=1,2,3,4$) components of the metric tensor, g_{ij}, their conjugate momenta, π^{ij}, and two other functions N and N_i:

$$N=(-{}^5g_{00})^{1/2},\quad N_i={}^5g_{0i}\ . \tag{57}$$

Also, t is a parameter which distinguishes the four-dimensional spacelike slices.

Varying N and N_i yields $C^0=0$ and $C^i=0$, a set of constraints on g_{ij} and π^{ij}. We do this and reduce the action to the form

$$I=(16\pi^{-1})\int\pi^{ij}\left[\frac{\partial g_{ij}}{\partial t}\right]d^5x \tag{58}$$

subject to the constraints $C^0=0$, and $C^i=0$, where

$$\begin{aligned}C^0&=-\sqrt{g}\,\{{}^4R+g^{-1}[\tfrac{1}{2}(\pi^k{}_k)^2-\pi^{ij}\pi_{ij}]\}\ ,\\ C^i&=-2\pi^{ij}{}_{;j}\ ,\end{aligned} \tag{59}$$

where $g=\det(g_{ij})$, ${}^4R$ is the scalar curvature of $t=\text{const}$ surfaces, and a semicolon means covariant differentiation on $t=\text{const}$ surfaces.

Because we are considering homogeneous universes we can integrate over the space variables. Our choice of differential forms which appear in the metric leads to

$$\int d^4x=(4\pi)^3\ . \tag{60}$$

Also,

$$\int\frac{\partial g_{ij}}{\partial t}dt=\int dg_{ij}\ . \tag{61}$$

So,

$$I=4\pi^2\int\pi^{ij}dg_{ij}\ . \tag{62}$$

It is convenient to parametrize the diagonal matrix $\pi^i{}_k$ as follows:

$$\begin{aligned}p_\alpha&=(8\pi^2)\pi^k{}_k\ ,\\ p^i{}_k&=(8\pi^2)(\pi^i{}_k-\tfrac{1}{4}\delta^i{}_k\pi^1{}_1)\ ,\end{aligned} \tag{63}$$

with

$$\begin{aligned}12p^1{}_1&=p_0+\sqrt{2}p_++\sqrt{6}p_-\ ,\\ 12p^2{}_2&=p_0+\sqrt{2}p_+-\sqrt{6}p_-\ ,\\ 12p^3{}_3&=p_0-2\sqrt{2}p_+\ ,\\ 12p^4{}_4&=-3p_0\ .\end{aligned} \tag{64}$$

The result is that the action and the C^0 constraint can be rewritten

$$I=\int p_0dB_0+p_+dB_++p_-dB_-+p_\alpha d\alpha$$

and (65)

$$C^0=0-3p_\alpha{}^2+p_0{}^2+p_+{}^2+p_-{}^2-768\pi^4e^{8\alpha}({}^4R)\ .$$

We note that p_0, p_+, and p_- are the momenta conjugate to B_0, B_+, and B_-. We choose $\alpha/\sqrt{3}$ as our canonical time coordinate. $H=\sqrt{3}p_\alpha$, being conjugate to the time, becomes our Hamiltonian.

We solve the C^0 constraint equation for p_α:

$$H^2=-3p_\alpha{}^2=p_0{}^2+p_+{}^2+p_-{}^2-768\pi^4e^{8\alpha}({}^4R)\ . \tag{66}$$

We can adjust the zero of α and redefine it so

$$768\pi^4e^{8\alpha}\to e^{8\alpha}\ .$$

We now define $V=1-e^{-2\alpha}({}^4R)$. Thus

$$H^2=p_0{}^2+p_+{}^2+p_-{}^2+e^{6\alpha}(V-1)\ . \tag{67}$$

H is our ADM Hamiltonian. We can now write out Hamilton's equations for this Hamiltonian:

$$\begin{aligned}&\frac{d}{d\alpha}B_0=\frac{\partial H}{\partial p_0},\quad \frac{d}{d\alpha}B_\pm=\frac{\partial H}{\partial p_\pm}\ ,\quad \frac{dH}{d\alpha}=\frac{\partial H}{\partial\alpha}\ ,\\ &\frac{d}{d\alpha}p_0=-\frac{\partial H}{\partial B_0},\quad \frac{d}{d\alpha}p_\pm=-\frac{\partial H}{\partial p_\pm}\ .\end{aligned} \tag{68}$$

The potential $V(B)$ is a combination of exponential functions which is determined by the structure constants of the homogeneous model in question. We have calculated these for the groups classified by Fee. These potentials are presented in Table II. The simplest case ($G0$) is when $(V-1)$ vanishes. Then, $H=(p_0{}^2+p_+{}^2+p_-{}^2)^{1/2}$, and the shape of the Universe is characterized by a free particle moving in a space spanned by the B_0, B_+, and B_- axes. In the general case, the Universe behaves as a point particle in the potential well V.

In the four-dimensional case, it has been noted that the qualitative nature of the universe in question is related to the shape of the equipotential surfaces for this potential well. In particular, the fact that the potentials corresponding to vacuum Bianchi types VIII and IX are closed determines that these types exhibit a chaotic behavior.[16] For the diagonal models the exact form of their potential is

$$\begin{aligned}V(B_-,B_+)=\ &1+\tfrac{2}{3}\exp(4B_+)[\cosh(4\sqrt{3}B_-)-1]\\ &+\tfrac{1}{3}\exp(-8B_+)\\ &\pm\tfrac{4}{3}\exp(-2B_+)\cosh(2\sqrt{3}B_-)\ ,\end{aligned} \tag{69}$$

where the plus sign corresponds to the Bianchi type-VIII case. The coordinates B_+ and B_- are analogous to the coordinates B_0, B_+, and B_- in the five-dimensional case. This potential has exponentially steep walls with equipotentials forming equilateral triangles in the (B_+,B_-) plane. For B_+ and B_- close to zero, the equipotentials look like circles. The corners of the triangle are not closed, but instead have thin channels leading off to infinity.

The chaotic behavior of these space-times can be ex-

TABLE II. Potentials (V) associated with the five-dimensional homogeneous cosmologies.

$G0$	1
$G1$	$1+\frac{1}{2}\exp(-10B_0+2\sqrt{2}B_+-2\sqrt{6}B_-)$
$G2$	$1+\frac{1}{2}[\exp(-2B_0+4\sqrt{2}B_++4\sqrt{6}B_-)+\exp(-10B_0+2\sqrt{2}B_+-2\sqrt{6}B_-)]$
$G3$	$1+2\exp(-2B_0+4\sqrt{2}B_+)+\frac{1}{2}\exp(-10B_0+2\sqrt{2}B_+-2\sqrt{6}B_-)$
$G4$	$1+2\exp(-2B_0)[\exp(4\sqrt{2}B_+)+\exp(2\sqrt{6}B_--2\sqrt{2}B_+)]$
$G5$	$1+6\exp(-2B_0+4\sqrt{2}B_+)-\exp(-2B_0-2\sqrt{2}B_++2\sqrt{6}B_-)$ $+\frac{1}{2}[\exp(-10B_0-4\sqrt{2}B_+)+\exp(6B_0+4\sqrt{6}B_-)]$
$G6$	$1+\frac{1}{2}[\exp(-2B_0+4\sqrt{2}B_++4\sqrt{6}B_-)+\exp(-10B_0+2\sqrt{2}B_+-2\sqrt{6}B_-)]$ $+12\exp(-2B_0+4\sqrt{2}B_+)$
$G7$	$1+(6+4P+2P^2)\exp(-2B_0+4\sqrt{2}B_+)+\frac{1}{2}\exp(-10B_0+2\sqrt{2}B_+-2\sqrt{6}B_-)$
$G8$	$1+(6P^2+2Q^2+2P+2Q+2PQ+2)\exp(-2B_0+4\sqrt{2}B_+)$
$G9$	$1+(6P^2+2Q^2+4PQ-1)\exp(-2B_0+4\sqrt{2}B_+)$ $+\frac{1}{2}[\exp(-10B_0+2\sqrt{2}B_+-2\sqrt{6}B_-)+\exp(6B_0+6\sqrt{2}B_++2\sqrt{6}B_-)]$
$G10$	$1+20\exp(-2B_0+4\sqrt{2}B_+)$ $+\frac{1}{2}[\exp(-10B_0-4\sqrt{2}B_+)+\exp(-2B_0+4\sqrt{2}B_++4\sqrt{6}B_-)]$
$G11$	$1+(6+10P+6P^2)\exp(-2B_0+4\sqrt{2}B_+)+\frac{1}{2}\exp(-10B_0-4\sqrt{2}B_+)$
$G12$	$1+(20P^2-1)\exp(-2B_0+4\sqrt{2}B_+)$ $+\frac{1}{2}\exp(-10B_0-4\sqrt{2}B_+)+\exp(-2B_0+4\sqrt{2}B_+)\cosh(4\sqrt{6}B_-)$
$G13$	$1+\exp(6B_0)[\cosh(4\sqrt{6}B_-)-1]+\frac{1}{2}\exp(-10B_0-4\sqrt{2}B_+)$ $-2\exp(-2B_0-2\sqrt{2}B_+)\cosh(2\sqrt{6}B_-)$
$G14$	$1+\exp(6B_0)[\cosh(4\sqrt{6}B)-1]+\frac{1}{2}\exp(-10B_0-4\sqrt{2}B_+)$ $+2\exp(-2B_0-2\sqrt{2}B_+)\cosh(2\sqrt{6}B_-)$

plained in the following manner. Within the Bianchi type-IX potential, the universe point first bounces against two potential walls until it enters the corner channel formed by these walls. One approximates the exponentially steep equipotentials as infinitely hard walls. Then the point leaves the corner to bounce against the walls corresponding to a different corner channel. This process repeats itself indefinitely. The chaotic behavior occurs because neighboring trajectories diverge as they are followed both forward and backward in time. This can only occur in models where the potential is closed, aside from the presence of the corner channels.

None of the five-dimensional homogeneous space-times in fact have the closed potential wells required for chaotic behavior. Cross sections of the $G1$, $G2$, and $G14$ potentials are depicted in Figs. 2—4. It is clear that closed potential wells, required for the manifestation of chaotic behavior, do not exist for five-dimensional homogeneous space-times as classified by Fee.

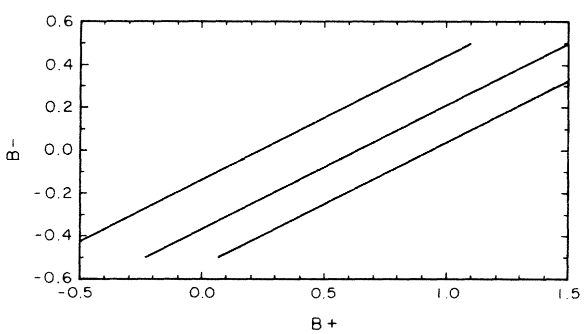

FIG. 2. Depicted here are the equipotential surfaces ($V=2$, $V=4$, and $V=8$) for the five-dimensional spacetime classified as $G1$. The two-dimensional figures represent three-dimensional potentials, sliced by the $B_0=0$ plane. Note that these potentials are all open, precluding the possibility of chaos. The nearest line to the bottom of the graph corresponds to $V=2$; the furthest to $V=8$.

For example, one can use this method to show that the $G13$ model, the five-dimensional mixmaster analogue, is not chaotic. One can rewrite the ADM Hamiltonian in this case as

$$2H=-p_\alpha{}^2+p_0{}^2+p_+{}^2+p_-{}^2 -(1-V')\exp(6\alpha-\sqrt{2}B_0)\ , \tag{70}$$

where

$$V'(B_-,B_+)=1+\exp(4B_+)[\cosh(4\sqrt{3}B_-)-1] -2\exp(2-B_+)\cosh(2\sqrt{3}B_-) +\tfrac{1}{2}\exp(-8B_+)\ .$$

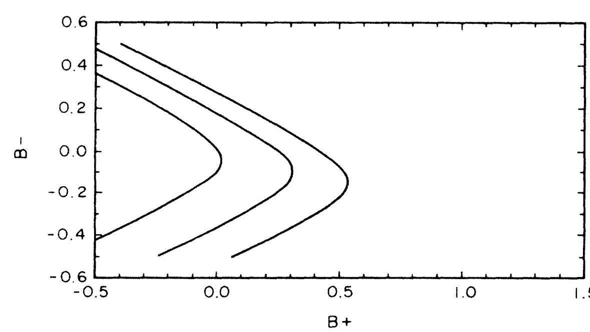

FIG. 3. The equipotential surfaces as in Fig. 2 for $G2$. The inner curve corresponds to $V=2$. The outer one corresponds to $V=8$.

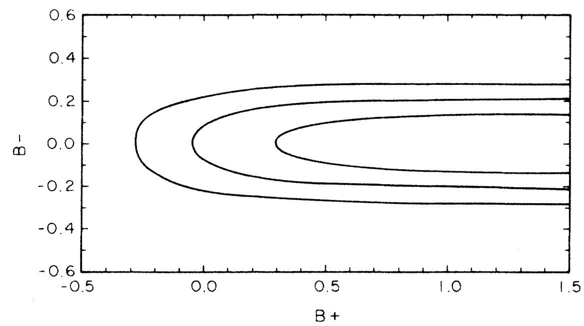

FIG. 4. The equipotential surfaces as in Fig. 2 for $G14$. The inner curve corresponds to $V=2$; the outer, to $V=8$.

V', the part of the potential that is dependent only on B_+ and B_-, is recognizable as the potential for the four-dimensional Bianchi type-IX model. The equipotentials in the B_+, B_- plane are triangular and closed, except for the corners where the potential increases exponentially. This can be seen by looking at the asymptotic form for V as $B_+ \to -\infty$:

$$V=\tfrac{1}{2}\exp(-8B_+) \ . \tag{71}$$

However, one can look at the Hamiltonian for this system and note that the full potential also is dependent on B_0. So, the momentum p_0 conjugate to B_0 is nonzero. In fact, in the $B_0 \to -\infty$ direction, the equipotential triangles become larger exponentially. Therefore, the full potential in the five-dimensional case is open. This precludes the existence of chaotic behavior.

It is likely that closed equipotentials wells will not be found for any higher-dimensional anisotropic cosmology. If this is the case, then chaos in cosmology is a dimension-specific phenomenon.

V. CONCLUSIONS

We have found several interesting properties of the class of solutions to the five-dimensional Einstein equations. This set of solutions possesses important qualitative differences from its four-dimensional counterpart.

For several of the Fee types, no diagonal vacuum solutions exist. It is increasingly more difficult to find diagonal solutions as one increases the number of dimensions of the Einstein equations. This is because of the fact that as the number of dimensions increases by one, the number of nontrivial Einstein equations increases by two, and the dimensionality of the set of structures constants also increases. This places an increasing number of restrictions on the scale factors, since the number of scale factors is equal to the number of dimensions. Eventually, some of these restrictions could contradict others, eliminating the possibility of a diagonal solution.

Most of the Fee types admit solutions which exhibit dimensional reduction in which three spatial dimensions expand, while the remaining one contracts.

The set of Fee types which approach isotropy appears to be a set of zero measure in the space of all possible five-dimensional cosmologies. The nine Fee types comprising the most general set of homogeneous universes, in that their sets of structure constants have the maximum number of degrees of freedom, have yet to be fully explored. A stability analysis of this set would indicate whether or not isotropy is possible at late cosmic times. Perhaps there is a set of finite measure in the space of all initial data which leads to a universe which conforms to our experimental limits of anisotropy, as determined by the measurement of the microwave background radiation? This question has been examined in the case of the four-dimensional Bianchi models.[10]

We have shown that none of the five-dimensional models are chaotic. This is a marked contrast with the four-dimensional case. We speculate that chaos is a dynamical property that is only possible in the $(3+1)$-dimensional case. Chaos only manifests itself in the case where, in a Hamiltonian ADM analysis, the Universe is a point in a closed potential well. This requires a certain symmetry of the set of structure constants specifying the universe type. For example, in the four-dimensional Bianchi type-IX case, the Einstein equations are invariant under a cyclic permutation of the axes. This symmetry is impossible for higher-dimensional models.

Studies of the Einstein equations for space-time of greater than four dimensions yield several benefits. They may be of use in the development of realistic higher-dimensional unified field theories. Failing that, they certainly aid in answering the question, "In what sense is our Universe, of three spatial dimensions, special among the set of universe of all possible numbers of dimension?"

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