1964

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JOURNAL OF MATHEMATICAL PHYSICS VOLUME 5, NUMBER 11 NOVEMBER 1964

Observables in Relativistic Quantum Mechanics*

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(Received 3 March 1964; final manuscript received 2 July 1964)

The conventional statement of statistical determinism is that "the expectation values of all (Heisenberg) observables are determined by the expectation values of the observables *at one time*." This requires that a full algebra of self-adjoint operators be in one-to-one correspondence with measurement procedures performed *at one time*. For instance, it requires that if two noncommuting observables p and q are defined at $t=0$, there should exist a measurement procedure at $t=0$ corresponding to $p+q$. No such procedure is known. The contrast between the positive assertion of the existence of certain laboratory procedures and the inability to describe them constitutes perhaps the weakest point of quantum mechanics. However, the conventional statement of statistical causality is shown to be untenable in a relativistic theory. This paper proposes a weaker form of causality which (1) uses measurements made within a truncated light cone rather than at one time for predictive purposes, and (2) which involves only strictly localized states, i.e., states which are vacuumlike outside a finite volume. Failure of the conventional causality statement implies that the set of quasilocal observables is not necessarily linear, i.e., if A and B are in a set, $A+B$ is not necessarily in it. This remark may open the way to a systematic inquiry into the problems of associating laboratory procedures to self-adjoint operators.

I. INTRODUCTION

THE fact that quantum mechanics is an incomplete theory is generally acknowledged and deplored by those who are interested in fundamental problems. Quantum mechanics asserts[1,2] that measurement procedures at one time are in one-to-one correspondence with an algebra of self-adjoint operators on Hilbert space, but it does not specify the procedures. As an example, assume that procedures for measuring the position q and the momentum p at the time $t = 0$ are known. Quantum mechanics asserts that there exist procedures performed at $t = 0$ which correspond to $p + q$. The assertion does not mean only that it is possible to design a procedure by which the sum of the expectation values $\langle p\rangle_\Psi + \langle q\rangle_\Psi$ is obtained for every state Ψ. If this were the whole assertion, the procedure could be trivially specified as an arithmetic addition of numbers obtained from many individual measurements of p and of q on samples of the ensemble Ψ. The assertion is that the same procedure should also yield the expectation values of $(p + q)^2$ and of other real-valued functions of the operator $p + q$. For this purpose, results of the measurement of $(p + q)$ on individual samples may be squared and averaged. One could, for instance, measure q and then p in rapid succession and consider the sum of the observed values as the value of $(p + q)$. However, the more accurately q is measured, the wider the statistical dispersion of subsequent values of p, until, in the limit, the measured value of p becomes entirely independent of the original state.

Also, two measurements of $(p + q)$ performed in rapid succession should give the same or almost the same value. These requirements, imposed by the theory on the apparatus, cannot be met by any known device. On the other hand, the sum of two commuting observables A and B may be defined simply as the arithmetic addition operation on the two procedures. Operationally, the test for commutativity is to determine if the expectation value $\langle A + B\rangle$ is independent of the order in which the measurements are performed.

Why is it necessary to maintain the stringent postulate in the face of obvious difficulties? What would the theory lose in predictive power if the postulate were dropped or weakened? It is shown that the usual assumption about the correspondence between operators and procedures is indispensable for the commonly accepted form of statistical causality (or determinism).[3] The assertion of causality is

(*A*) *"The expectation values of the observables measured at one time (on a spacelike hypersurface) de-*

* This work performed under the auspices of the U. S. Atomic Energy Commission.

[1] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, England 1947), p. 26.

[2] J. von Neumann, *Mathematische Grundlagen der Quanten-Mechanik* (Dover Publications, Inc., New York, 1943), p. 167.

[3] No precise distinction between the two words seems to enjoy universal acceptance.

termine the expectation values of all observables at later times."

This seems to be a minimal substitute for classical causality, and it is understandable that one goes far to save it.

We shall see that in the context of the general principles of quantum mechanics the causality (A) requires, in effect, that the observables on one spacelike hypersurface $t = 0$ form a linear set. In particular, for a complete set of dynamical variables $p(0)$, $q(0)$, the linear combination $p(0) + q(0)$ must also be an observable at $t = 0$. The dilemma apparently is this: either we must find procedures for measuring such quantities as $(p + q)$ at $t = 0$, or we must abandon what seems to be a reasonably minimal form of causality. Yet, as we shall show, the statement (A) conflicts with the combination of (1) the relativistic principle of signal propagation with a finite velocity and (2) well-established nonclassical effects such as measurability of parity. Therefore, one must accept a weaker form of statistical causality which does not refer to such all-inclusive categories as "observables at time t anywhere in the universe" but, more modestly and realistically, to quasilocal observables [Sec. IV, Statement (C)].

The weaker form of causality does not demand that observables at one time form a linear set, and hence relieves us of the burden of trying to design extraordinary experimental procedures to satisfy the requirements of a theory. This result opens the way to a systematic investigation of the relation between laboratory procedures and self-adjoint operators on Hilbert space.

II. CONSEQUENCES OF CONVENTIONAL CAUSALITY

A measurement procedure in a space–time volume V or spacelike hyperplane S is a set of instructions and apparatus for an operation carried out within V or S; that is, all interaction between the apparatus and the system takes place within V or S.

The assumption that such procedures exist clearly requires some extrapolative idealization. If a measuring instrument begins to interact with the system at the time t in a space volume v, the instrument must have been brought there previously, thus disturbing the system. To justify this assumption, it must be asserted that the interaction previous to t can be minimized to any desired degree.

In Secs. II–IV we are not interested in correlation measurements, i.e., subsequent measurements on the same sample of an ensemble. We may assume that each sample of the ensemble is destroyed or discarded after the measurement. However, we do not assume that *each* measurement is instantaneous, and we classify observables by the time interval of measurement, i.e., the interval beginning with the interaction between apparatus and object and ending at the moment when the necessary information is stored.

Many measurement procedures are equivalent in that they give identical results for all ensembles. An equivalence class of measurement procedures in V (or S) are called an observable in V (or S). Different observables may have identical expectation values for all ensembles, e.g., the momentum of a free particle, measured at different times. This defines an equivalence class of observables which, following Dirac, we call a "dynamical variable." Self-adjoint operators on Hilbert space may be considered as images of observables in a many-to-one mapping, or as images of dynamical variables in a one-to-one mapping.

We follow the conventional assumption to the extent that the set E of dynamical variables is assumed to form a normed linear space so that the set of all observables is closed under addition. For example, if $p(0)$ and $q(0)$ are observables, then $q(0) + p(0)$ may not be an observable at $t = 0$; but it is *an* observable. This is a much weaker assumption than isomorphism between dynamical variables and observables at one time. For instance, for a free particle [$q(t) = q(0) + pt$], the Heisenberg operator q measured at the time $t = 1$ is equal to $p(0) + q(0)$. In other words, the equivalence class of the dynamical variable $q(0) + p(0)$ may not include an observable at $t = 0$, but it does include one at $t = 1$. In the remainder of this section, we consider only observables at one time (or on a spacelike hyperplane).

Let G_t be the set of dynamical variables observable at t, i.e., G_t consists of those dynamical variables whose equivalence class includes an observable at t. To an ensemble ρ, one associates expectation values of dynamical variables $\langle A\rangle_\rho$ $(A \in E)$. They form a positive linear functional $f_\rho(A)$ on the dynamical variables. According to Statement (A), the expectation values of the particular dynamical variables B $(B \in G_t)$ determine all expectation values. In other words, if two ensembles have identical expectation values for all dynamical variables $B \in G_t$, then they also have identical expectation values for all dynamical variables. That is,

$$f_\rho(B) = f_{\rho'}(B) \qquad (B \in G_t) \tag{2.1}$$

implies

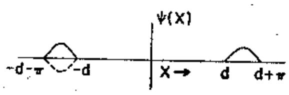

FIG. 1. Wavefunction of a one-dimensional particle with positive (full line) and negative (dotted line) reflection parity.

$$f_\rho(A) = f_{\rho'}(A) \qquad (A \in E). \tag{2.2}$$

The vanishing of a linear functional $f_\rho - f_{\rho'}$ in G_t implies its vanishing on the whole set E; such a subset G_t is called *total*. *Note added in proof*: It is assumed that every linear functional can be represented as the difference between two positive linear functionals. This assumption is justified only for certain topologies of the space of dynamical variables.

We need two definitions in order to state the consequences of this postulate.

Definition 1. A subset G is dense in E if, for each $y \in E$, there exists a Cauchy sequence of elements $X_n \in G$ so that $X_n \to y$.

Definition 2. A subset G is fundamental if the set of all linear combinations of elements of G is dense in E.

The condition for G_t being total [i.e., the condition for the postulate (A)] is then given by the theorem:[4]

Theorem. A subset G is total if and only if it is fundamental.

For the purpose of designing measurement procedures, we can go farther. The knowledge of the expectation values of observables A_i is equivalent to the knowledge of the expectation values of all linear combinations of the A_i. Also, there is no physical distinction between a procedure for obtaining a mean value and one which allows approximating it to any desired degree. Hence:

Physically, a fundamental set of dynamical variables is equivalent to the whole set.

To summarize, the postulate (A), together with the general principles of quantum mechanics, requires a one-to-one correspondence between the set of observables at one time and the set of all dynamical variables.

In an attempt to avoid the unpleasant consequences, one might weaken the statement of causality in an obvious way by requiring knowledge of expectation values of observables in a spacelike *slab* of finite thickness in the time dimension [Statement (B)].[5]

[4] S. Banach, *Théorie des Opérations Linéaires* (Hafner Publishing Company, New York, 1932), p. 58.
[5] R. Haag and B. Schroer, J. Math. Phys. **3**, 249 (1963).

While this weakening constitutes a further departure from the idea that the present determines the future, it does not seem unreasonable as long as the thickness of the slab is small. This idea will not be pursued in the present paper since the next section will show that neither Statement (A) nor Statement (B) is tenable in a relativistic theory.

III. THE FAILURE OF STATISTICAL CAUSALITY IN RELATIVITY

The finite velocity of signal propagation imposes severe restrictions on the possibilities of the measuring apparatus. Since the measuring instruments are macroscopic, it is sufficient to apply the basic principles of classical relativity to their operation.

Consider a space volume v at a time t, and let S_v be the set of all observables that can be measured in v at time t. If v' is another nonintersecting volume, a measurement in v cannot influence one in v' at the time t. That is, any instantaneous measurement by an instrument which occupies both v and v' supplies no more information than could be obtained by simultaneous separate measurements in v and in v'. The same conclusion obviously holds if V and V' are space–time volumes which are spacelike with respect to each other, i.e., if V includes only points that are separated by spacelike intervals from all points of V'.

Consider a state that is vacuumlike everywhere except in two congruent disjoint volumes v and v', i.e., the expectation values of all quasilocal observables at $t = 0$ outside of v and v' are those of the vacuum state. The remaining information is supplied by quasilocal observables in the space–time volumes V and V' which include v and v'. We may assume V and V' to be spacelike, with respect to each other. As an example, consider a one-dimensional one-particle system with two states described by the wavefunctions $\psi(x)$ (Fig. 1):

$$\psi = \begin{cases} 0 \quad \text{except for} \quad d < |x| < d + \pi, \\ \frac{1}{2}(2)^{\frac{1}{2}} \sin(x - d) \quad \text{for} \quad d < x < d + \pi, \\ \pm\frac{1}{2}(2)^{\frac{1}{2}} \sin(-x + d) \ \ \text{for} \ \ -d - \pi < x < -d. \end{cases} \tag{3.1}$$

Instantaneous observations in the two segments $d < |x| < d + \pi$ cannot distinguish between the two signs. By the principle of finite signal propagation, the time necessary to obtain additional information cannot be made arbitrarily small. If a photon is used for the purpose of *comparing* the physical situations in the two segments, the minimal time for obtaining information would be $2(d + \pi)/c$.

On the other hand, we know that the reflection operator, defined by

$$R\psi(x) = \psi(-x) \tag{3.2}$$

corresponds to an observable. The two functions in Eq. (3.1) are eigenfunctions of R with parity (eigenvalue) ± 1, and there are known methods for determination of parity.

If, more generally, observations in a finite time interval are admitted, the same conclusions hold if the space–time volumes

$$-d-\pi < x < -d, \qquad |t| < \Delta t$$

and

$$d < x < d+\pi, \qquad |t| < \Delta t$$

are spacelike with respect to each other. For any finite timelike thickness Δt, there are states (characterized by ψ in our example) whose observable properties cannot be determined by an observation in the timelike slice.

We conclude that the strong causality [Statement (A)] as well as the slightly weakened form (B) are untenable in relativistic quantum mechanics.

IV. WEAK CAUSALITY

A strictly localized ensemble ρ_v has the property that at $t=0$ the expectation values of all observables are vacuumlike outside the space volume v. More precisely, if w is a space volume entirely outside v, and A_w a quasilocal observable at $t=0$ in w, then the expectation value $\langle A_w\rangle_{\rho_v}$ is equal to the vacuum expectation value $\langle A_w\rangle_\Omega$ of this observable. According to Sec. III, there exist observables whose expectation values are not functions of the instantaneous expectation values $\langle A\rangle_{\rho_v}$. Consider, however, a four-dimensional cone defined as follows. Let R be the radius of the smallest sphere that contains v. Then this sphere and the hypersurface consisting of all light rays from the surface of the sphere to its center defines a space–time cone $C(v)$, shown in Fig. 2, such that observations in C can ascertain any "phase relation" between parts of the physical system in v. Without contradicting either the relativistic principle of finite signal velocity or well-established results of quantum mechanics, we can state a weaker form of causality:

(C) *For a strictly localized ensemble ρ_v in the space volume v, the expectation values of the observables A_C in the corresponding space–time cone $C(v)$ determine all expectation values.*

It might appear, at first, that Statement (C) is

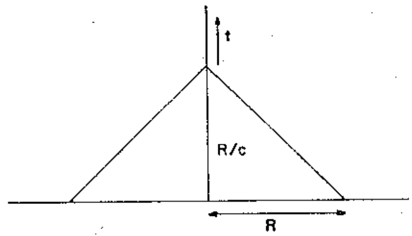

FIG. 2. Weak causality: For a state strictly localized within a sphere of radius R_i, a determining set of observations must include durations of R/c.

too weak to be useful as a substitute for (A) since the strictly localized ensembles are a very special class of ensembles or states. However, the concept of a physical system is meaningful only to the extent that it is not influenced by other parts of the universe which are left out in considering the system. If, nevertheless, the system is idealized so that it extends everywhere, then we must make the assumption that nothing else exists, i.e., that the expectation values of quasilocal observables are vacuumlike at sufficiently large distances. Statement (C) seems to be adequate not because most ensembles are strictly localized but because the only way to deal with actual ensembles is to approximate them by strictly localized ones. In contrast to Statements (A) and (B), (C) evidently does not require that the set of observables associated with a precise instant should be closed under addition, and thereby relieves the theorist of a heavy burden.

Another consequence of the principle of finite signal velocity is that a collection of strictly localized ensembles $\{\rho_v\}$ for a fixed volume v is invariant under operators that are images of the corresponding set of quasilocal observables $\{A_v\}$. Indeed, according to the principles of quantum mechanics the vector $A_v\Psi_v/||A_v\Psi_v||$ is the state created immediately after an instantaneous measurement A_v. If this state differed from the vacuum state outside of v, a signal would be transmitted instantaneously from v to other space points.

This remark can serve to confirm the impossibility of determining the phase δ in a state of the type considered in Sec. III, viz.

$$\Phi = \Psi_v + e^{i\delta}\Psi_{v'}$$

by instantaneous measurements if v and v' are disjoint simultaneous space volumes. Clearly,

$$(\Psi_{v'},\Psi_v) = 0,$$

and according to our previous remark

$$(\Psi_{v'}, A\Psi_{v'}) = 0$$

whether A is in $\{A_v\}$ or $\{A_{v'}\}$. Hence, for any observable A,

$$(\Phi, A\Phi) = (\Psi_v, A\Psi_v) + (\Psi_{v'}, A\Psi_{v'})$$

and the cross term always vanishes, so that no instantaneous information about the phase is available.

V. THE PROJECTION AXIOM

The weakening of classical determinism in quantum mechanics is of two kinds: either the statements refer to all observables and all states but to the ensemble rather than the individual sample, or they refer to some observations on some states and successive observations on one sample. The latter cases are realized by a special kind of measurement procedure called a "procedure of the first kind,"[6] which is aptly described as *filtering*. A filter selects a subset of an original ensemble, and some unambiguous predictions can be made with respect to each sample of such a subset. Let us consider the restrictions that relativity imposes on these predictions.

One of von Neumann's postulates is the projection axiom (M)[7]: "If the observable R is measured on a system twice in succession, both observations yield the same value." Clearly, this form of the statement must be taken with a grain of salt. Margenau[8] has pointed out that in the overwhelming majority of measurements the system under observation is destroyed; it or its parts become permanently attached to the measuring apparatus. In the spirit of Pauli,[6] a more literal version would preface the sentence by "In every equivalence class of procedures belonging to the observable R, there exists one such that . . . ."

Is Axiom M necessary at all? It is argued here that at least in some modified form "Axiom M" is both physically desirable and indispensable. In classical physics the immediate repetition of an observation confirms the first result. This fact is tacitly accepted as the basis of any science. If it were not so, could one speak of objectively true events at all? Since quantum mechanics must agree with classical physics in some limit, quantum mechanics must surely include some statement with predictive claim on successive measurements of individual samples. What could the statement be?

Von Neumann points out that there are, *a priori*, three possible forms of causality or acausality in relation to the repetition test. (The words "confirmability" or "objectivity" would perhaps be more felicitous than "causality.") Given a repetition, (1) the first and second results could be statistically independent, (2) the first result could have a statistical dispersal, but the second be each time identical with the first, or (3) both results could be uniquely determined by the initial state.

The third case is that of classical mechanics; the first would come close to denying the existence of any objective observation, and hence of natural science. There remains the second case which is embodied in Axiom M—and perhaps a fourth possibility, namely that the results of the second measurement could be statistically correlated to the first. The principle of simplicity impels us to choose the second rather than the fourth possibility unless there is definite evidence against the former.

The point in which von Neumann's axiom needs revision (in addition to the minor restriction made above) is the time after which a confirmatory repetition can be made. As we have seen, in relativistic quantum mechanics some observables that are indispensable for prediction cannot be measured instantly, i.e., there is an inevitable delay between the beginning of the interaction and the recording of the information. It is now shown that there is equally an inevitable delay before the second measurement can confirm the first result.

In discussing the time sequence of measurements, it is convenient to think of a retrospective analysis of measurements completed in the distant past, rather than of a theory to be applied to experiments in actual progress. The first advantage of this view is that the use of probability in the sense of a rational judgment on the basis of existing and, ordinarily, incomplete evidence never arises; the only kind of probability involved is the relative frequency of past events. The second advantage of the retrospective view is that the question of signal velocity between recording devices never arises. It must be remembered that, literally speaking, a *prediction* is not possible even in classical relativistic physics, since the time necessary to communicate information from local observing devices to a central predictor would be precisely as long as the time for which the theoretical predictive ability claims validity, viz. $t = \Delta/c$, where Δ is the distance between the most distant of the simultaneous recording devices. Instead, by prediction we mean the establishment of a functional relation between observations recorded at different times, all of them in the distant past with respect to the time at which the verification is made.

[6] W. Pauli, *Handbuch der Physik* (Springer-Verlag, Berlin, 1933), p. 152.
[7] Reference 2, p. 177.
[8] H. Margenau, Phys. Rev. 49, 240 (1936).

We consider the procedures of the first kind which measure the parity of a sample at two different times in such a way that the first and second measurements have identical results.

Resonance scattering of light provides such a procedure for some systems. Let there be two energy "ground" eigenstates with opposite parity, and let the state be a coherent superposition of the two (which are assumed to be nearly degenerate). If there is an excited energy eigenstate of known parity (say $+1$), then resonance scattering for sufficiently long wavelength is possible only with parity change of the system. If the energy spread of the photon covers the energy difference between the two ground states and the excited states then the system is certain to be in the state with parity -1 after resonance of the photon has been observed. The question is now: What is the smallest time between the beginning of the interaction between system and photon and its cessation? To simplify the question, think of a system which is initially in the negative-parity state, and ask for the time at which the wavefunction of the combination (system and photon) becomes a product function with the negative-parity eigenfunction as one factor.

Consider an electrodynamic system (such as a positronium) consisting of two particles localized approximately in small volumes v and v' with a large distance between them. Intuitively, the answer to our question is then the following. In order to be sensitive to the parity of the state, the photon has to be scattered by one particle (say, in v) and run to v' to be rescattered—or vice versa. The smallest time for such a process is evidently d/c.

A more quantitative estimate may be derived by elementary perturbation theory. Dyson's operator[9] $U(t, t_0)$ for finite times can be expanded and the terms transformed in the usual manner. The result of the contractions can be represented by the usual diagrams. The relevant fourth-order diagrams are shown in Fig. 3; it is the interference of these two terms that is sensitive to parity. The rules for the evaluation of the diagrams differ from the usual ones only in that the integration over the coordinates of points 1 and 4 is omitted. The result is the probability amplitude for a process in which a photon reaches the system at t_1 and leaves it at t_4. In order to obtain the probability for a resonance scattering from a ground state, the resulting matrix element would have to be integrated with respect to the initial and final coordinates of the particles. Similarly, an integration over some localized wave packet of photons would have to be performed on the final and initial coordinates of the photon.

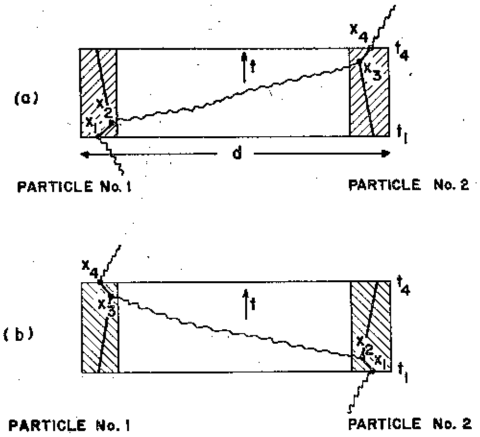

Fig. 3. Feynman diagrams for resonance scattering. The photon is absorbed and re-emitted by one particle, then absorbed and re-emitted by the other particle.

The usual evaluation of the diagrams exhibits the function $D_F(x_3 - x_2)$, where the time coordinates of x_3 and x_2 cannot differ by more than $t_4 - t_1$ while the space coordinates differ approximately by d. Since the function D_F decreases rapidly outside the light cone, the matrix element is negligible unless the time $t_4 - t_1$ is larger than d/c. The measurement begins when the photon interacts with particle No. 1 (or 2) and is repeatable after it has interacted with particle 2 (or 1) in diagram (a) [or (b)]. In the intermediate period, one of the particles is in an excited state and the total system is clearly not in the ground state, so that an additional photon would not be scattered in the same manner.

Only a particular class of fourth-order diagrams has been considered, and one may ask why others should not contribute to the measurement. Physically, the reason is that the photon energy has been chosen for resonance (i.e., so that Thompson scattering, Compton scattering, etc., are negligible), but mathematically this cannot be shown from perturbation since the excited intermediate state is not obtainable by perturbation.

It has thus been shown that relativity imposes a time delay between a measurement of the first kind and the subsequent confirmatory observation. Therefore, the term "immediate repetition" in Axiom M must be replaced by the phrase "repetition after the time Δt that characterizes the space–time

[9] See, for example, S. S. Schweber, H. A. Bethe, and F. De Hoffmann, *Mesons and Fields* (Row, Peterson and Co., Evanston, Illinois, 1955), Vol. I.

volume V assigned to the observables." This leads to the modified axiom:

(M') *In every equivalence class of procedures belonging to an observable* A_V, *there exists a procedure such that for all systems whose Hamiltonian commutes with* A_V, *its repetition after the time* Δt *gives the same result as the first measurement, where the interval* Δt *is the largest timelike interval in* V.

ACKNOWLEDGMENTS

The authors gratefully acknowledge helpful discussions with H. Araki, J. M. Cook, and R. Haag.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 5, NUMBER 11 NOVEMBER 1964

Relativistic Coulomb Scattering of Electrons*

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A simple and useful relation between the Coulomb amplitudes F and G (in Mott's notation) is derived and F and G are evaluated analytically up to α^4 terms for arbitrary $q = \alpha/\beta$. These results are valid for all angles, but are particularly useful at small angles. The general analytic behavior of F and G in the variable $x = \sin \frac{1}{2}\theta$ is discussed. The method is applicable to higher-order terms (α^6 and up). A double integral representation of F is also derived by using the Sommerfeld–Watson transformation. This integral representation exhibits the dependence on α, q, and θ separately.

1. INTRODUCTION

THE solutions for the relativistic scattering of electrons in a Coulomb field were first obtained by Mott[1] in the form of partial wave amplitudes. These amplitudes were expressed as functions of the two parameters α and q, where $\alpha = Z/137$, $q = \alpha/\beta$, and $\beta = v/c$. Attempts to sum the partial wave series analytically were successful only in powers of α (with β considered to be of order 1).[2,3] The most recent of these attempts[3] led to expressions for the Coulomb amplitudes F and G (in Mott's notation) accurate to order α^2 and α^3 respectively, with extremely complicated coefficients which were functions of β and $x = \sin \frac{1}{2}\theta$.

We have obtained a simple and useful relation between the Coulomb amplitudes F and G, and have succeeded in summing the partial wave series in powers of α^2 for arbitrary q, up to and including the terms in α^4. This organization of the expansion appears to be simpler and more natural than that in simultaneous power of α and $q^{2,3}$, since the major complexity of the latter comes from expansion of the Coulomb phase factor, exp $(2iq \ln x)$, in powers of q. In our expansion the result is separated into two terms, one of which contains the phase factor exp $(2iq \ln x)$, the other of which does not. These results are then analytic in the variable x, apart from the Coulomb phase factor. This separation is similar to that given by Drell and Pratt[4] for $\beta = 1$.

Our results are related to those of Rosen,[5] and of Fradkin, Weber, and Hammer.[6] Rosen derived a double-integral representation of the coefficients of powers of α^2. Fradkin *et al.* derived a similar expansion in terms of a two-parameter function $T(\theta, q)$ up to α^2. We have evaluated these coefficients as convergent expansions in powers of x, which are most useful in the small-angle region (near $x = 0$). In addition, the method is applicable for the α^6 and higher terms, although the algebra is tedious and has not been carried out.

For completeness we have also obtained a double-integral representation of the Coulomb amplitudes, in which the dependence on α, q, and θ is exhibited in separate factors.

Applications of the considerations in the present paper to physical problems have been considered

* Supported in part by the National Science Foundation.
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