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Algebraic Description of Motion

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An algebraic definition of time differentiation is presented and used to relate independent measurements of position and velocity. With this, students can grasp certain essential physical, geometric, and algebraic properties of motion and differentiation before undertaking the study of limits.

I. INTRODUCTION

Since the end of the seventeenth century, when Newton and Leibniz began the development of differential calculus, the velocity of an object at one time has usually been defined as the limit of its average velocity $\Delta x/\Delta t$ as the time interval Δt approaches zero. This definition has become so familiar to many of us that certain physical, mathematical, and pedagogical difficulties with it are often overlooked. Among these are:

(1) Physical measurements of the velocity at one time cannot be based directly on this definition since as Δt decreases sufficiently, the determination of average velocity by successive position measurements gives decreasing precision due to experimental, thermal, and quantum limitations.

(2) The physical significance of velocity as an independent physical quantity and the possibility of measuring it directly are obscured.

(3) While the algebraic and geometric aspects of differentiation commonly used in physics problems can be deduced from a definition in terms of limits, they need not be. There are important epistemological and pedagogical reasons for making no stronger assumptions than are needed and for choosing definitions which relate as closely as possible to actual physical processes and typical computations concerning them.

(4) Most scientists, let alone first-year physics students, are not sufficiently familiar with the local topological properties of the continuum to appreciate the full implications of the deceptively simple assumption that the limit of $\Delta x/\Delta t$ exists as $\Delta t \to 0$. The existence of this limit is usually taken for granted or justified intuitively even though it is a vast extrapolation beyond experience to aspects of motion over time and distance intervals on a sub-nuclear scale.

Other criticisms of this definition of velocity range from the philosophical ones in Berkeley's *Analyst*[1] to pedagogical ones in Saletan's *Velocity without Limits*.[2] While Saletan defines velocity without reference to limits, his definition does use the usual ordering of real numbers from which all their topological properties and hence the usual limits could be defined.

In this paper, we propose an alternative algebraic definition of differentiation which makes no use of limits or other topological concepts and which does not share the difficulties listed here for the usual one. While the basic mathematical ideas are standard algebraic ones, their use in physics seems to have been overlooked.

II. MEASUREMENT OF VELOCITY

While length and time have been chosen as primary quantities in terms of which kinematic ones can be defined, this is a result of historical and technical developments and is not logically necessary. Velocity could be chosen as a primary quantity to replace length or time, as is electric current in emu and SI units for the precisely analogous electrical case.

By putting position and velocity measurements on a more equal footing, we can better study the relations between them and derive either from the other, rather than being bound to the view that velocity is necessarily derived from position.

Some methods for measuring velocity directly which are of considerable scientific and technological importance are based on the following velocity dependent effects:

(1) Centripetal acceleration of a rotating mass—used for governors on old steam engines.

(2) Bernoulli pressure drop in a moving fluid—used to measure the air speed of some airplanes.

(3) Force between a magnet and a conducting loop in relative motion—used in automobile speedometers.

(4) Doppler frequency shift—used for determining interstellar and intergalactic velocities as well as in radar measurements of the velocities of nearby planets, satellites, and terrestrial objects.

(5) Cerenkov radiation—used to measure the velocity of charged particles travelling through matter at speeds greater than c/n, where c is the velocity of light in vacuum and n is the refractive index of the matter through which the particle moves.

When measuring time-dependent velocities, each actual measurement is a weighted average of the instantaneous velocity over the response time of the measurement apparatus. A better approximation to the instantaneous velocity can be achieved by shortening the response time or by mathematical processing of the data. These features are common to measurements of all time-dependent quantities—position, for example, as well as velocity—and give no reason for considering velocity to be a derived quantity or less directly measurable than others.

In the absence of restrictions on the time dependence of forces on a particle, no finite sequence of position measurements, no matter how short the time duration between them, can provide any strict bounds at all, let alone precise values for the velocity at any time. This is the case even with the assumption that the position varies as a "smooth" (e.g., polynomial) function of time. It is useful therefore to divide the process of relating position to velocity into three parts. In the first, position measurements are used to determine a smooth interpolating function for the time dependence of position. In the second part, the velocity as a smooth function of time is obtained from this interpolating function by time differentiation to be defined algebraically in the next section. Finally, independent measurements of velocity at any time are compared with this derivative.

III. AN ALGEBRAIC DEFINITION OF TANGENTS AND DIFFERENTIATION

Methods which make no use of limits have been known since antiquity for constructing tangents to circles and other conics. It seems to be less widely known that finite or algebraic means exist for constructing tangents to any algebraic curve, i.e., one whose abscissa and ordinate satisfy an algebraic equation. Our definition of differentiation and its geometric interpretation is based on the following property of polynomials[3].

Theorem 1: For every polynomial q in time, there is just one polynomial function q' with $q(t)=q(a)+(t-a)q'(a)+(t-a)^2f(t,a)$ for some polynomial function f and for every t and a.

Proof: By substituting $(t-a)+a$ for t and expanding, any polynomial q in t can be expressed as a polynomial in $t-a$ with coefficients which are polynomials in a. We obtain the expression for $q(t)$ needed to prove the theorem by then factoring $(t-a)^2$ from terms of second and higher degree.

Definitions: The *time-derivative* q' of any polynomial q in time is the unique one specified in theorem 1. The *tangent* at time $t=a$ to the graph of q is the graph of the first degree polynomial, $q(a)+(t-a)q'(a)$.

The equality between the value of the time derivative and the slope of the tangent follows directly from these definitions.

Theorem 2: Time differentiation has the properties

(i) $(t-a)'=1$ for any number a,

(ii) $(p+q)'=p'+q'$ and

(iii) $(pq)'=p'q+pq'$ for any polynomials p and q. Conversely, if p^* is the image of p under any map of polynomials into polynomials satisfying these three properties, then $p^*=p'$, the time derivative of p, for all p.

The proof follows directly from the definition of the time derivative and the fact that two polynomials in $t-a$ are equal if and only if their corresponding coefficients are all equal. Actually, only the coefficients of the first power of $t-a$ need be considered.

Theorem 2 establishes that properties i, ii, and iii give an equivalent algebraic definition of the time-derivative of polynomials. These algebraic properties are the ones most frequently used when carrying out differentiations. Properties ii and iii alone define a *derivation* on the ring of polynomials[4] and there is just one derivation satisfying property i. Of these three properties, the one that many find least intuitive is the product rule, iii. Indeed, one of the sounder parts of Berkeley's criticism of Newton's fluxions, at least by current standards, concerned just this property, as Hamilton noted in a letter to the mathematician DeMorgan.[5] Its geometric significance can be seen by considering the growth of a rectangle whose sides p and q are functions of time. If the orientation of its sides and the position of one vertex remains fixed, the trajectory of the opposite vertex divides the rectangle into two parts whose areas change at the rates $p'q$ and pq'. Thus the total area changes at the rate $(pq)'=p'q+pq'$.

As a mathematical exercise to demonstrate conclusively that no hidden use has been made of limits or other topological assumptions in this definition of differentiation and tangents, it may be instructive to substitute a finite number field, for example, the field of integers modulo a prime p, for the coefficients, arguments, and values of the polynomials.[3] The graphs of polynomials and their tangents then each consists of just a finite number of points, though for large p they exhibit many features of the usual ones.

IV. CURVE FITTING

Having given an algebraic definition of differentiation, we return to the first and third parts of the process of relating position to velocity measurements mentioned at the end of Sec. II, i.e., we use the results of N position measurements, each made with finite precision, to determine the best values of up to N parameters for a family of interpolating functions. From these, an estimate for both the mean value and the standard deviation for the velocity at any time can be obtained by differentiation. These can then be compared with velocity measurements.

For a specific and useful example of this procedure, we consider a situation in which it is believed reasonable to assume that the acceleration remains constant over the period of time that measurements are made. Then, at least three position measurements are needed to determine the values of the three coefficients of the second degree polynomial function which gives the best fit to the position data. For reasons of simplicity and symmetry, assume that four measurements are made at times $-3a/2$, $-a/2$, $a/2$ and $3a/2$ with results $x_{-3a/2}$, $x_{-a/2}$, $x_{a/2}$, and $x_{3a/2}$ and that they are all made with the same precision. The best estimate for the mean value and standard deviation for the velocity v_0 at $t=0$ is then

$$v_0=(10a)^{-1}(3x_{3a/2}+x_{a/2}-x_{-a/2}-3x_{-3a/2}$$
$$\pm|x_{3a/2}-3x_{a/2}+3x_{-a/2}-x_{-3a/2}|).$$

This result is obtained by standard methods[6] based on the assumption that the measure of goodness of fit, chi-squared, is to have its mean value, the number of measurements less the number of parameters, which in this case is $4-3=1$.

V. CONCLUSION

The velocity of an object at one time is considered an independent physical quantity directly measurable by such methods as those listed in Sec. II. The process of relating independent position and velocity measurements can be divided into three parts—fitting a polynomial function to the position measurements, differentiating this polynomial by the algebraic means presented in Sec. III, and then comparing the result with velocity measurements. An example is given in Sec. IV for a procedure to estimate the velocity at one time and its standard deviation based on four position measurements.

By applying these methods both to actual position and velocity measurements as well as in computations, students can grasp the essential physical, algebraic, and geometric aspects of differentiation before they undertake any study of limit processes.

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[1] G. Berkeley, *Analyst: Or a Discourse Addressed to an Infidel Mathematician* (London, 1734), reprinted in *A Source Book in Mathematics*, edited by D. Smith (Dover, New York, 1959), pp. 627–634.

[2] E. Saletan, Am. J. Phys. **41**, 792 (1973).

[3] Physics students may find helpful a computationally-oriented discussion of these algebraric concepts in W. W. Sawyer, *A Concrete Approach to Abstract Algebra* (Freeman, San Francisco, CA, 1959), pp. 33–70.

[4] O. Zariski and P. Samuel, *Commutative Algebra* (Van Nostrand, New York, 1958), pp. 120–131.

[5] R. P. Graves, *Life of Sir William Rowan Hamilton* (Dublin, 1882–89), Vol. 3, p. 569.

[6] J. Mathews and R. Walker, *Mathematical Methods of Physics* (Benjamin, New York, 1970), 2nd ed., pp. 387–396.