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Birefringent gravitational waves and the consistency check of inflation

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In this work we show that the gravitational Chern-Simons term, aside from being a key ingredient in inflationary baryogenesis, modifies superhorizon gravitational waves produced during inflation. We compute the super-Hubble gravitational power spectrum in the slow-roll approximation and show that its overall amplitude is modified while its spectral index remains unchanged (at leading order in the slow-roll parameters). Then, we calculate the correction to the tensor to scalar ratio, T/S. We find a correction of T/S which is dependent on $\mathcal{N}$ (more precisely quadratic in $\mathcal{N}$), the parameter characterizing the amplitude of the Chern-Simons terms. In a stringy embedding of the leptogenesis mechanism, $\mathcal{N}$ is the ratio between the Planck scale and the fundamental string scale. Thus, in principle, we provide a direct probe of leptogenesis due to stringy dynamics in the cosmic microwave background. However, we demonstrate that the corresponding correction of T/S is in fact very small and not observable in the regime where our calculations are valid. To obtain a sizable effect, we argue that a nonlinear calculation is necessary.

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I. INTRODUCTION

Cosmic baryogenesis stands as one of the unresolved problems of particle cosmology. Most models address baryogenesis after the inflationary epoch. Recently the authors of Ref. [1] demonstrated that the baryon asymmetry can be generated during inflation from gravity waves. In this model the lepton number was generated by a quantum expectation value of the Chern-Simons density from ultraviolet (UV), birefringent gravitational waves during the inflationary epoch. In a subsequent paper the authors showed that this model can be embedded in string theory, in a model independent manner, through the Green-Schwarz mechanism [2]. In the stringy embedding there was a huge enhancement of the lepton asymmetry due to a hierarchy in the fundamental string scale and the four dimensional Planck scale.

Reference [3] was the first to study the cosmological consequences of a Chern-Simons term and to notice that this implies parity violation in the cosmic microwave background (CMB) which ultimately leads to a birefringence in gravitational waves. This line of reasoning was then pursued by several other authors [4–8]. In this paper, we shall study the superhorizon power spectrum and tensor to scalar ratio of scalar and tensor birefringent perturbations produced during inflation. Specifically, we study the spectrum of superhorizon gravity waves whose UV counterparts were responsible for leptogenesis. Is it possible to see a signature of the leptogenesis mechanism in the superhorizon power spectrum? To address this question we shall derive the tensor to scalar ratio and show that it contains a direct signature of the leptogenesis mechanism which occurred in the UV. Furthermore we show that the scalar to tensor ratio contains the string scale in a model independent way and is in an observable window to this physics.

The paper is organized as follows. In Sec. II we derive the equations for the gravitational waves in the presence of the Chern-Simons term. In Sec. III we provide the exact solutions at various scales and derive the power spectrum as well as the corrected tensor to scalar ratio. In Sec. IV we relate this modified tensor to scalar ratio to the stringy embedding of gravitational leptogenesis and we conclude with some open issues concerning a consistent quantization and further directions.

II. BASIC EQUATIONS

The starting point of inflationary leptogenesis is the Einstein-Hilbert action coupled to the gravitational Chern-Simons term, which is necessarily present in string theory. This last term can be written as

$$S_{\rm CS} = \frac{1}{8\kappa}\int {\rm d}^4x f(\phi) R\wedge R, \tag{1}$$

where $\kappa \equiv 8\pi/m_{\rm Pl}^2$, $m_{\rm Pl}$ being the Planck mass. We proceed to linearize the Einstein-Hilbert action with the Chern-Simons term in a Friedmann-Lemaître-Robertson-Walker (FLRW) background in the presence of tensor perturbations (i.e. in presence of gravitational waves). The corresponding metric tensor takes the form (assuming that the spacelike sections are flat)

$$ds^2 = a^2(\eta)[-{\rm d}\eta^2 + (\delta_{ij} + h_{ij}){\rm d}x^i{\rm d}x^j], \tag{2}$$

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with h_{ij} being a transverse and traceless tensor, i.e. $\delta^{ij}h_{ij}=0$, $\partial^j h_{ij}=0$ and $a(\eta)$, the FLRW scale factor, being a function of the conformal time η. Because of the symmetries of the FLRW metric, the inflaton field ϕ in Eq. (1) is also a function of the conformal time only.

Expanding the action up to second order in the gravitational waves tensor h_{ij} (which is necessary in order to obtain first order equations of motion), after lengthy but straightforward calculations, one obtains the following expression:

$$ {}^{(2)}S_{\rm GW}=\frac{1}{8\kappa}\int \mathrm{d}^4x\{a^2(\eta)[(h^i{}_j)'(h^j{}_i)'-(\partial_k h^i{}_j)(\partial^k h^j{}_i)] - f'\epsilon^{ijk}[(h^q{}_i)'(\partial_j h_{kq})'-(\partial^r h^q{}_i)\partial_j\partial_r h_{kq}]\}, \tag{3}$$

where a prime stands for a derivative with respect to conformal time and $\epsilon^{ijk}\equiv\epsilon^{0ijk}$, $\epsilon^{\mu\nu\tau\sigma}$ being the totally antisymmetric tensor. In the above expression, one recognizes the standard (i.e. Einstein-Hilbert) expression of the perturbed action (first term between squared brackets) while the term proportional to f' represents the correction coming from the Chern-Simons contribution. Varying this action with respect to the gravitational waves tensor, one obtains the first order equation of motion which reads

$$ (h^j{}_i)''+2\frac{a'}{a}(h^j{}_i)'-\partial_k\partial^k h^j{}_i+\frac{1}{a^2}\epsilon^{pjk}[f''(\partial_p h_{ki})' + f'(\partial_p h_{ki})''-f'\partial_p\partial^r\partial_r h_{ki}]=0. \tag{4}$$

Next, following Ref. [4], we define the tensor D_{ij} by the following equation:

$$ D_{ij}\equiv h''_{ij}+2\frac{a'}{a}h'_{ij}-\partial_k\partial^k h_{ij}, \tag{5}$$

and, then, the equation of motion takes the form

$$ D^j{}_i+\frac{1}{a^2}\epsilon^{pjk}[(f''-2\mathcal{H}f')\partial_p h'_{ki}+f'\partial_p D_{ki}]=0, \tag{6}$$

where we have defined $\mathcal{H}\equiv a'/a$. This equation is similar to Eqs. (11) and (12) of Ref. [4], except that we have written the equation in terms of the conformal time rather than in terms of the cosmic time.

The next step consists in going to the Fourier space. For this purpose, we write the metric tensor as

$$ h_{ij}(\eta,\mathbf{x})=\frac{1}{(2\pi)^{3/2}}\int \mathrm{d}\mathbf{k}\sum_{s=1}^{2}p^s_{ij}(\mathbf{k})h^s_{\mathbf{k}}(\eta)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}. \tag{7}$$

In the above expression, $p^s_{ij}(\mathbf{k})$ is the linear polarization tensor ($s=1,2$ corresponds to $s=+,\times$). Concretely, if the wave vector is written in polar coordinates as $\mathbf{k}/k=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$, then two vectors perpendicular to $\mathbf{k}$ are given by $\mathbf{e}_1=(\sin\varphi,-\cos\varphi,0)$ for the first vector and $\mathbf{e}_2=(\cos\theta\cos\varphi,\cos\theta\sin\varphi,-\sin\theta)$ for the second vector but only if $\theta<\pi/2$. If $\theta>\pi/2$, i.e. if the wave vector points to the bottom, the expression of the second vector should in fact read $\mathbf{e}_2=-(\cos\theta\cos\varphi,\cos\theta\sin\varphi,-\sin\theta)$. It is also interesting to mention how these quantities transform under the change $\mathbf{k}\to-\mathbf{k}$. It is easy to see that this corresponds to the transformation $(\theta,\varphi)\to(\pi-\theta,\varphi+\pi)$. Then, we have $\mathbf{e}_1\to-\mathbf{e}_1$ and $\mathbf{e}_2\to-\mathbf{e}_2$. Finally, the polarization tensor can be written as

$$ p^1_{ij}=(\mathbf{e}_1)_i(\mathbf{e}_1)_j-(\mathbf{e}_2)_i(\mathbf{e}_2)_j, \tag{8}$$

$$ p^2_{ij}=(\mathbf{e}_1)_i(\mathbf{e}_2)_j+(\mathbf{e}_1)_j(\mathbf{e}_2)_i. \tag{9}$$

Because of the properties of the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ established above, it is easy to check that $p^s_{ij}(-\mathbf{k})=p^s_{ij}(\mathbf{k})$ and $p^s_{ij}(\mathbf{k})p^{ijs'}(\mathbf{k})=2\delta^{ss'}$. Using these properties and the fact that h_{ij} is real, $h_{ij}=h^*_{ij}$, one can also establish that

$$ (h^s_{\mathbf{k}})^*=h^s_{-\mathbf{k}}, \qquad s=+,\times. \tag{10}$$

The next step consists in defining two other states of polarization, the so-called right and left polarization states. The corresponding polarization tensors are given by

$$ p^{\rm R}_{ij}\equiv\frac{1}{\sqrt{2}}(p^1_{ij}+ip^2_{ij}), \tag{11}$$

$$ p^{\rm L}_{ij}\equiv\frac{1}{\sqrt{2}}(p^1_{ij}-ip^2_{ij})=(p^{\rm R}_{ij})^*. \tag{12}$$

From the above expressions, using the properties of the linear polarization tensors, one can show that

$$ p^{\rm R}_{ij}(\mathbf{k})p^{ij{\rm R}}(\mathbf{k})=p^{\rm L}_{ij}(\mathbf{k})p^{ij{\rm L}}(\mathbf{k})=0, \tag{13}$$

$$ p^{\rm R}_{ij}(\mathbf{k})p^{ij{\rm L}}(\mathbf{k})=2. \tag{14}$$

These expressions are of course valid only if the polarization tensors are evaluated for the same wave number. We also have $p^s_{ij}(\mathbf{k})=p^s_{ij}(-\mathbf{k})$ with $s=\mathrm{R},\mathrm{L}$. Then, using the expression of the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, it is easy to show that

$$ \frac{k_p}{k}\epsilon^{mpj}p^s_{ij}=\mp i\lambda^s(p^m{}_i)^s, \qquad s=\mathrm{R},\mathrm{L}, \tag{15}$$

where $\lambda^{\rm R}=+1$ and $\lambda^{\rm L}=-1$ and where the upper sign refers to $\theta<\pi/2$ while the lower one refers to $\theta>\pi/2$.

We are now in a position where one can rewrite the gravitational waves tensor in terms of the left and right polarization states. This gives

$$ h_{ij}(\eta,\mathbf{x})=\frac{1}{(2\pi)^{3/2}}\int \mathrm{d}\mathbf{k}\sum_{s=\mathrm{R,L}}p^s_{ij}(\mathbf{k})h^s_{\mathbf{k}}(\eta)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}, \tag{16}$$

where we have introduced the definitions

$$ h^{\rm R}_{\mathbf{k}}\equiv\frac{1}{\sqrt{2}}(h^1_{\mathbf{k}}-ih^2_{\mathbf{k}}), \qquad h^{\rm L}_{\mathbf{k}}\equiv\frac{1}{\sqrt{2}}(h^1_{\mathbf{k}}+ih^2_{\mathbf{k}}). \tag{17}$$

Let us notice that it is straightforward to demonstrate that $h^{\rm R}_{-\mathbf{k}} = (h^{\rm L}_{\mathbf{k}})^*$ and $h^{\rm L}_{-\mathbf{k}} = (h^{\rm R}_{\mathbf{k}})^*$.

The next step consists in introducing the new expansion of the gravitational waves tensor given by Eq. (16) into the equation of motion and in using Eq. (15) to arrive at a new form of the equation of motion. One obtains

$$\left(1 - \lambda^s k \frac{f'}{a^2}\right)(h^s_{\mathbf{k}})'' + \left(2\mathcal{H} - \lambda^s k \frac{f''}{a^2}\right)(h^s_{\mathbf{k}})' + \left(1 - \lambda^s k \frac{f'}{a^2}\right)k^2 h^s_{\mathbf{k}} = 0, \qquad s = \mathrm{R, L}. \quad (18)$$

Finally, the last step consists in introducing the quantity z_s defined by

$$z_s(\eta, \mathbf{k}) \equiv a(\eta)\sqrt{1 - \lambda^s k \frac{f'}{a^2}} \quad (19)$$

and the new amplitude $\mu^s_{\mathbf{k}}(\eta)$ defined by $\mu^s_{\mathbf{k}} \equiv z_s h^s_{\mathbf{k}}$. Then, the equation of motion for $\mu^s_{\mathbf{k}}$ has the traditional form of the equation of motion for a parametric oscillator, namely

$$(\mu^s_{\mathbf{k}})'' + \left(k^2 - \frac{z_s''}{z_s}\right)\mu^s_{\mathbf{k}} = 0. \quad (20)$$

The effective potential z_s''/z_s depends on time, on polarization (birefringence) but also on the wave number which is an important difference with respect to the standard case where the effective potential depends on conformal time only. This equation has been derived for the first time in Ref. [4]; see Eq. (15) of that paper. However, in Ref. [4], it is also assumed that the effective potential takes the form $z_s''/z_s = n_s/\eta^2$ where n_s is a constant. In particular, one notices that, with this ansatz, the scale dependence of the effective potential has disappeared. This permits one to find simple solutions in terms of Bessel functions. However, we will see that, in the present context, the effective potential is different and more complicated.

III. GRAVITATIONAL WAVES POWER SPECTRUM IN THE SLOW-ROLL APPROXIMATION

A. The effective potential

Let us now calculate the effective potential explicitly. Using the formulas established previously, one obtains that the exact expression of the potential can be written as

$$\frac{z_s''}{z_s} = \frac{a''}{a} - \mathcal{H}\lambda^s k \frac{(f'/a^2)'}{1 - \lambda^s k (f'/a^2)} - \frac{\lambda^s k}{2}\frac{(f'/a^2)''}{1 - \lambda^s k (f'/a^2)} - \frac{1}{4}(\lambda^s k)^2 \frac{[(f'/a^2)']^2}{[1 - \lambda^s k (f'/a^2)]^2}. \quad (21)$$

To go further, we need to postulate the function f. Following Ref. [1], we choose

$$f = \frac{\mathcal{N}}{16\pi^2 M_{\rm Pl}^2}\frac{\phi}{M_{\rm Pl}}, \quad (22)$$

where $M_{\rm Pl} \equiv m_{\rm Pl}/\sqrt{8\pi}$ is the reduced Planck mass and $\mathcal{N}$ a number that we discuss in more detail in the last section and that can be related to the string scale. With this definition f/a^2 is dimensionless, as it should, if the scale factor has the dimension of a length (which is our convention). In terms of the slow-roll parameters $\epsilon \equiv -\dot{H}/H^2$, $\delta \equiv -\ddot{\phi}/(H\dot{\phi})$ and $\xi \equiv (\dot{\epsilon} - \dot{\delta})/H$ (a dot means a derivative with respect to cosmic time), we have at leading order in the slow-roll parameters, see also Ref. [9]

$$a(\eta) = \ell_0(-\eta)^{-1-\epsilon}, \qquad \phi' \simeq -M_{\rm Pl}\mathcal{H}\sqrt{2\epsilon}. \quad (23)$$

From this expression, one deduces that (at leading order in the slow-roll parameters)

$$\frac{f'}{a^2} = -\frac{\mathcal{N}}{16\pi^2 M_{\rm Pl}^2}\frac{\mathcal{H}}{a^2}\sqrt{2\epsilon} \simeq \frac{\mathcal{N}}{16\pi^2}\left(\frac{H_{\rm inf}}{M_{\rm Pl}}\right)^2\sqrt{2\epsilon}\,\eta \quad (24)$$

because $\mathcal{H} \simeq -(1+\epsilon)/\eta$. From that expression, one arrives at the two following formulas which are useful in order to calculate the effective potential:

$$\left(\frac{f'}{a^2}\right)' = \frac{\mathcal{N}}{16\pi^2 M_{\rm Pl}^2}\frac{\mathcal{H}^2}{a^2}(1+\delta)\sqrt{2\epsilon} \quad (25)$$

$$\simeq \frac{\mathcal{N}}{16\pi^2}\left(\frac{H_{\rm inf}}{M_{\rm Pl}}\right)^2\sqrt{2\epsilon} + \mathcal{O}(\epsilon^{3/2}), \quad (26)$$

and

$$\left(\frac{f'}{a^2}\right)'' = \frac{\mathcal{N}}{16\pi^2 M_{\rm Pl}^2}\frac{\mathcal{H}^3}{a^2}(-\epsilon - \delta - 3\epsilon\delta + 2\epsilon^2 - \delta^2 - \xi) \times \sqrt{2\epsilon} \quad (27)$$

$$\simeq \frac{\mathcal{N}}{16\pi^2}\left(\frac{H_{\rm inf}}{M_{\rm Pl}}\right)^2\frac{1}{\eta}(\epsilon + \delta)\sqrt{2\epsilon} + \mathcal{O}(\epsilon^{5/2}). \quad (28)$$

Inserting the above equations into the formula giving the potential, namely, Eq. (21), one obtains

$$\frac{z_s''}{z_s} \simeq \frac{2+3\epsilon}{\eta^2} - \lambda^s \frac{k}{|\eta|}\frac{\mathcal{N}}{16\pi^2}\left(\frac{H_{\rm inf}}{M_{\rm Pl}}\right)^2\sqrt{2\epsilon}\left[1 - \lambda^s\frac{\mathcal{N}}{16\pi^2} \times \left(\frac{H_{\rm inf}}{M_{\rm Pl}}\right)^2\sqrt{2\epsilon}(k\eta)\right]^{-1} - \frac{k^2}{4}\frac{\mathcal{N}^2}{256\pi^4}\left(\frac{H_{\rm inf}}{M_{\rm Pl}}\right)^4 \times (2\epsilon)\left[1 - \lambda^s\frac{\mathcal{N}}{16\pi^2}\left(\frac{H_{\rm inf}}{M_{\rm Pl}}\right)^2\sqrt{2\epsilon}(k\eta)\right]^{-2}, \quad (29)$$

where we have ignored subdominant term in the slow-roll parameters. It is important to notice that, in order to obtain the above equation, we have never expanded a term like $1 - \lambda^s k(f'/a^2)$ in the slow-roll parameters. We notice the presence of k in the numerator of the second term. This is in full agreement with Ref. [10] where it has been noticed that a term like $1/|\eta|$ in the effective potential necessarily implies a new characteristic scale. Here, the characterized scale defined in Ref. [10] could be written as (at this level, the two situations are not yet totally equivalent because the above effective potential is not exactly similar to the potential studied in Ref. [10]. This will be the case below.)

$$k_{\rm C} \equiv k\frac{\mathcal{N}}{32\pi^2}\left(\frac{H_{\rm inf}}{M_{\rm Pl}}\right)^2\sqrt{2\epsilon} = k\frac{\Theta}{16}, \tag{30}$$

where we have defined Θ by the following relation [see Eq. (13) in Ref. [1]]:

$$\Theta \equiv \frac{\mathcal{N}}{2\pi^2}\left(\frac{H}{M_{\rm Pl}}\right)^2\sqrt{2\epsilon}. \tag{31}$$

In the present context, somehow, the characteristic scale $k_{\rm C}$ "depends on the scale" (i.e. on k). However, we see that the large-scale limit, as defined in Ref. [10] i.e. $k \ll k_{\rm C}$, corresponds in the present context to the condition $\Theta/16 \gg 1$. The only way to satisfy this condition is to have a large $\mathcal{N}$ which could compensate the smallness of $H/M_{\rm Pl}$ and of the slow-roll parameter.

For convenience, we now introduce the variable x defined by $x \equiv \Theta k\eta/8 < 0$. Then, the equation of motion takes the form

$$\frac{{\rm d}^2\mu}{{\rm d}x^2} + \left[\frac{64}{\Theta^2} - f_{\rm R,L}(x)\right]\mu = 0, \tag{32}$$

with

$$f_{\rm R}(x) = \frac{2+3\epsilon}{x^2} + \frac{1}{x(1-x)} - \frac{1}{4}\frac{1}{(1-x)^2}, \tag{33}$$

$$f_{\rm L}(x) = \frac{2+3\epsilon}{x^2} - \frac{1}{x(1+x)} - \frac{1}{4}\frac{1}{(1+x)^2}. \tag{34}$$

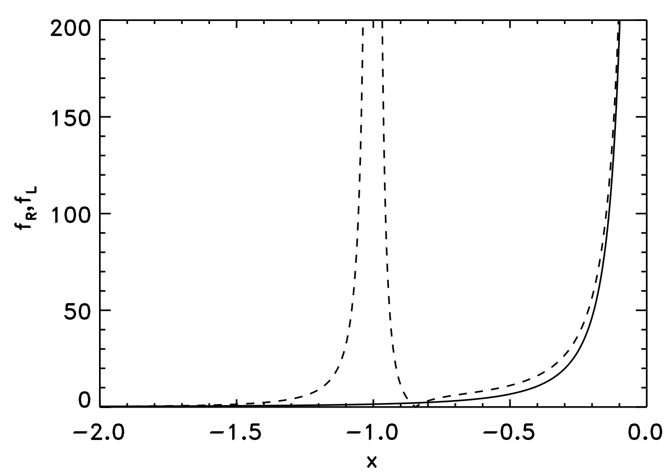

FIG. 1. Effective potential for the two states of polarization (solid line for the right polarization state and dashed line for the left polarization state). At $x = -1$ or $\eta = -8/(k\Theta)$, the effective potential $f_{\rm L}(x)$ blows up. For $x > -1$, the slight difference between $f_{\rm L}(x)$ and $f_{\rm R}(x)$ mathematically originates from the term $\lambda^s/[x(1-\lambda^s x)]$ in z_s''/z_s and, physically, from the phenomenon of birefringence. As $x \to 0$, the standard term $(2 + 3\epsilon)/x^2$ dominates. Since this term does not depend on the polarization state, one has $f_{\rm R}(x) \to f_{\rm L}(x)$.

The functions $f_{\rm R}$ and $f_{\rm L}$ are represented in Fig. 1. From this figure, the different behavior of the two states of polarization is apparent. The L mode (dashed line) undergoes a "kick" at $x = -1$ where the effective potential blows up. At the same point the potential of the R mode is perfectly regular (solid line). Therefore, we expect the R mode function to propagate smoothly through $x = -1$ while the behavior of the L mode function can be more problematic. We now turn to this question in more detail.

B. Solutions to the mode equation in the vicinity of the divergence

Let us now study the equation of motion for the left mode in the vicinity of $x \simeq -1$. It is easy to check that a very good approximation of the potential is

$$f_{\rm L}(x) \simeq \frac{1}{(1+x)} - \frac{1}{4}\frac{1}{(1+x)^2}. \tag{35}$$

In fact the approximation is good even far from $x \simeq -1$ provided $x < -1$ since, on small scales, i.e. in the limit $k\eta \to +\infty$, we have $z_s''/z_s \to 0$. In the limit, the solution can be written as

$$\mu_{\bf k}^s(\eta) \simeq A_1^s(k){\rm e}^{-ik\eta} + A_2^s(k){\rm e}^{ik\eta}, \tag{36}$$

where $A_1^s(k)$ and $A_2^s(k)$ are two constants that are fixed by the choice of the initial conditions. Usually, one requires that, on sub-Hubble scales

$$\mu_{\bf k}^s(\eta) = -\frac{4\sqrt{\pi}\ell_{\rm Pl}}{\sqrt{2k}}{\rm e}^{-ik(\eta-\eta_i)}. \tag{37}$$

This prescription completely fixes the coefficients $A_1^s(k)$ and $A_2^s(k)$ which read

$$A_1^s(k) = -\frac{4\sqrt{\pi}\ell_{\rm Pl}}{\sqrt{2k}}{\rm e}^{ik\eta_i}, \qquad A_2^s(k) = 0. \tag{38}$$

In the above equation, $\ell_{\rm Pl}$ is the Planck length and η_i is some initial time at the beginning of inflation. The knowledge of this time is not important since it will drop out from the final result.

With the potential given by Eq. (35), the equation of motion can be solved exactly. Indeed, if we define $\tau \equiv 16i(1+x)/\Theta$ then the equation of motion takes the form

$$\frac{{\rm d}^2\mu_{\bf k}^{\rm L}}{{\rm d}\tau^2} + \left[-\frac{1}{4} + \frac{i\Theta}{16\tau} + \frac{1}{4\tau^2}\right]\mu_{\bf k}^{\rm L} = 0. \tag{39}$$

This is the well-known Whittaker equation, see Eq. (9.220.1) of Ref. [11]. The corresponding solution, correctly normalized, see Eqs. (38), reads

$$\mu_{\mathbf{k}}^{\mathrm{L}} = -\frac{4\sqrt{\pi}\ell_{\mathrm{Pl}}}{\sqrt{2k}} \mathrm{e}^{ik\eta_i} \mathrm{e}^{-\pi\Theta/32} W_{i\Theta/16,0}\left[\frac{16i(1+x)}{\Theta}\right], \quad (40)$$

where $W_{\kappa,\mu}(z)$ is the Whittaker function.

Let us now study how the mode function behaves when $x \to -1$. The Whittaker function can be expressed in terms of the confluent hypergeometric function, see Eq. (13.1.33) of Ref. [12]. One obtains

$$\mu_{\mathbf{k}}^{\mathrm{L}} = -\frac{4\sqrt{\pi}\ell_{\mathrm{Pl}}}{\sqrt{2k}} \mathrm{e}^{ik\eta_i} \mathrm{e}^{-\pi\Theta/32} \mathrm{e}^{-8i(1+x)/\Theta} \sqrt{\frac{16i}{\Theta}(1+x)} \times U\left[\frac{1}{2} - i\frac{\Theta}{16}, 1, \frac{16i}{\Theta}(1+x)\right], \quad (41)$$

where $U(a, b, z)$ is the above-mentioned confluent hypergeometric function. Using Eq. (13.5.9) of Ref. [12] which says that, when $z \to 0$, $U(a, b, z) \to -[\ln z + \Psi(a)]/\Gamma(a)$, where $\Gamma(z)$ is the Euler's integral of the second kind and where $\Psi(z) \equiv d\ln\Gamma(z)/dz$, see Ref. [11], one deduces that

$$\mu_{\mathbf{k}}^{\mathrm{L}} \underset{x\to -1}{\longrightarrow} \sqrt{1+x}\ln(1+x). \quad (42)$$

But what really matters is not the intermediate variable $\mu_{\mathbf{k}}^{\mathrm{L}}$ but in fact the amplitude of the gravitational waves itself given by $h_{\mathbf{k}}^{\mathrm{L}} \equiv \mu_{\mathbf{k}}^{\mathrm{L}}/z_{\mathrm{L}}(\eta)$, see Eq. (19). Since $z_{\mathrm{L}} \propto \sqrt{1+x}$, one obtains

$$h_{\mathbf{k}}^{\mathrm{L}} \underset{x\to -1}{\longrightarrow} \frac{1}{a(\eta)} \ln(x+1). \quad (43)$$

The conclusion is that the amplitude of the mode $(\mathbf{k}, \mathrm{L})$ blows up at the time corresponding to $x = -1$, that is to say at the time $\eta_{\mathrm{div}}(k)$ defined by

$$\eta_{\mathrm{div}}(k) \equiv -\frac{8}{k\Theta}. \quad (44)$$

At this point the linear theory of cosmological perturbations breaks down and becomes nonlinear.

An important feature of η_{div} is that it is scale dependent. This means that the physical wavelength of the Fourier modes $\lambda = (2\pi/k)a(\eta)$, at time $\eta = \eta_{\mathrm{div}}$, are all equal to the same physical length. Explicitly, one has

$$\frac{\lambda(\eta_{\mathrm{div}})}{\ell_{\mathrm{Pl}}} = \frac{\pi}{4}\left(\frac{H_{\mathrm{inf}}}{m_{\mathrm{Pl}}}\right)^{-1}\Theta. \quad (45)$$

Somehow, this is reminiscent of one of the possible formulations of the trans-Planckian problem of inflation [13,14] where it is postulated that a mode of comoving wave number k is "created" when its physical wavelength equals a given new fundamental scale in the theory (the idea being to test the robustness of the inflationary predictions to short distance modifications of the theory; therefore, it is typical in this context to consider that the new scale is the Planck length, see Ref. [13] for more details). It is then easy to show that the "time of creation" is inversely proportional to k as it is the case for η_{div}, see, in particular, the fifth paper in Ref. [14]. As a consequence, we see from Eq. (45) that Θ defines in fact a new scale the value of which depends on the inflation scale but also on the string scale since we will see that the string scale is hidden into the number $\mathcal{N}$ which participates to the definition of Θ, see Eq. (31). A possible way out to the question of the divergence would be to push the problem in the trans-Planckian regime. From the above equation, this means that the parameter Θ should satisfy

$$\Theta \lesssim \frac{H_{\mathrm{inf}}}{m_{\mathrm{Pl}}}. \quad (46)$$

Therefore, this boils down to a quite stringent constraint on Θ, typically $\Theta \lesssim 10^{-5}$. Unfortunately, we will see in the following that, for such small values of Θ, the modifications on large scales are not observable. If one wants to consider larger values of Θ, it seems that much more refined (i.e. nonlinear) calculations are necessary. This calculation is obviously beyond the scope of the present paper which is just exploratory.

Let us conclude this subsection by stressing the fact that, *a priori*, trans-Planckian effects do not play a deep role in the Chern-Simons theory under considerations in this work. Here, it is merely a technical trick which allows to avoid the nonlinear regime and to adopt the common assumption that the Fourier modes emerge from the trans-Planckian region in the vacuum state. But clearly, if $\Theta > 10^{-5}$, then the nonlinear calculation is in principle feasible without any trans-Planckian considerations.

C. Solutions to the mode equation on very large scales

Let us now study what happens on very large scales, i.e. in the limit where x vanishes. In the situation, the effective potentials can be very well approximated by the following equations:

$$f_s(x) \simeq \frac{2+3\epsilon}{x^2} + \frac{\lambda^s}{x} - \frac{1}{4}, \quad (47)$$

where we remind the reader that $\lambda^{\mathrm{R}} = +1$ and $\lambda^{\mathrm{L}} = -1$. Birefringence enters this equation via the term proportional to $1/x$ which changes its sign according to the polarization state under considerations. The term proportional to $1/x^2$ is the standard slow-roll term. The corresponding equation of motion takes the form

$$\frac{\mathrm{d}^2\mu_{\mathbf{k}}^s}{\mathrm{d}x^2} + \left[\frac{64}{\Theta^2} + \frac{1}{4} - \frac{\lambda^s}{x} - \frac{2+3\epsilon}{x^2}\right]\mu_{\mathbf{k}}^s = 0. \quad (48)$$

Once again, we have to deal with a Whittaker equation. In fact this equation (and the corresponding power spectrum) has been studied in detail in Ref. [10], see Eq. (8), and the corresponding power spectrum has been derived in that reference, see Eq. (15). Therefore, in the present paper we can use the results obtained in Ref. [10] and follow the procedure utilized in that reference. Let us introduce the new definitions

$$y \equiv i\sqrt{1+\frac{256}{\Theta^2}}x, \qquad \kappa \equiv \frac{i\lambda^s}{\sqrt{1+256/\Theta^2}}, \qquad \xi \equiv \frac{3}{2}+\epsilon. \tag{49}$$

In the following, we will consider that $256/\Theta^2 \gg 1$ (see the discussion at the end of the previous subsection) and, as a consequence, will simply approximate $\sqrt{1+256/\Theta^2}$ by $16/\Theta$. With the new definitions taken into account, the equation of motion takes the form

$$\frac{d^2\mu_{\mathbf{k}}^s}{dy^2}+\left[-\frac{1}{4}+\frac{\kappa}{y}+\left(\frac{1}{4}-\xi^2\right)\frac{1}{y^2}\right]\mu_{\mathbf{k}}^s = 0, \tag{50}$$

which is again the Whittaker equation. The situation is exactly similar to the one studied around Eq. (12) of Ref. [10]. The exact general solution to this equation is given in terms of Whittaker functions

$$\mu_{\mathbf{k}}^s(\eta) = C_1^s(k)W_{\kappa,\xi}(y)+C_2^s(k)W_{-\kappa,\xi}(-y), \tag{51}$$

where $C_1^s(k)$ and $C_2^s(k)$ are two constants fixed by the choice of the initial conditions.

As discussed in the preceding subsection, we will fix the initial conditions in the region $-1 < x \ll 0$ which is free of divergences. In this regime, the only natural choice that we have is to postulate a plane wave. This is equivalent to postulating that the nonlinear phenomena occurring around the divergence of the effective potential, provided they happen in the trans-Planckian region, will not affect the standard choice of the initial conditions in the region $x > -1$. Somehow, this is the same assumption that is made in the standard inflationary scenario. Indeed, despite the fact that the modes of astrophysical interest today originate from the trans-Planckian region [13], the vacuum is assumed to be the correct initial state. Let us stress, however, that a possible weakness of the above comparison is that, in the case of the trans-Planckian problem of inflation [13,14], one can show that the final result can be robust to changes in the short distance physics [13,14] (under some conditions, i.e. adiabatic evolution of the Fourier modes). In the present context, however, it is more difficult to imagine that the nonlinearities will not affect the initial conditions. On the other hand, in the absence of second-order calculations and as a first approach to the problem, this seems to be quite reasonable. As shown in Ref. [10], see Eqs. (14), this choice amounts to

$$C_1^s(k) = -\frac{4\sqrt{\pi}\ell_{\rm Pl}}{\sqrt{2k}}e^{iq\eta_i}\exp\left(-\frac{\lambda^s\pi\Theta}{32}\right), \tag{52}$$

$$C_2^s(k) = 0, \tag{53}$$

where we have used the fact that, see Eq. (9.227) of Ref. [11], $\lim_{|y|\to+\infty}W_{\kappa,\xi}(y) = e^{-y/2}y^{\kappa}$. The sign of the argument in the exponential depends on the polarization state considered, as expected. We conclude that the solution to the mode equation on very large scales is now known explicitly.

D. The power spectrum

Usually, the power spectrum is given by the two-point correlation function calculated in the vacuum state. Another way to calculate the same quantity is to view it as a classical spatial average. Since a fully consistent quantum formulation of the present theory is not yet available, we adopt the second point of view. Therefore, the two-point correlation function can be written as

$$\langle h_{ij}(\eta,\mathbf{x})h^{ij}(\eta,\mathbf{x})\rangle = \frac{1}{V}\int d\mathbf{x}\,h_{ij}(\eta,\mathbf{x})h^{ij}(\eta,\mathbf{x}), \tag{54}$$

with $V = \int d\mathbf{x}$ is the total volume. Using the properties of the polarization tensor, straightforward calculations show that

$$\langle h_{ij}(\eta,\mathbf{x})h^{ij}(\eta,\mathbf{x})\rangle = \frac{1}{\pi^2}\sum_{s={\rm L,R}}\int_0^{+\infty}\frac{dk}{k}k^3|h_{\mathbf{k}}^s|^2, \tag{55}$$

from which we deduce the power spectrum

$$k^3P_h^s(k) = \frac{k^3}{\pi^2}\left|\frac{\mu_{\mathbf{k}}^s}{a(\eta)\sqrt{1-\lambda^s kf'/a^2}}\right|^2. \tag{56}$$

Let us notice that, usually, the power spectrum is proportional to the factor $2k^3/\pi^2$. Here we do not have the factor 2 because we consider the two states of polarization separately (i.e. usually, these two states are summed and produce the factor 2).

A priori, using the solution obtained in the previous subsection, we can calculate the spectrum exactly in terms of the Whittaker function. But only the spectrum on large scales is needed and in this regime one has (for details, see Ref. [10])

$$k^3P_h^s = \frac{16}{\pi}\frac{\ell_{\rm Pl}^2}{\ell_0^2}\frac{k^{-2\epsilon}}{2^{2\xi}}\frac{\Gamma^2(2\xi)}{|\Gamma(1/2+\xi-i\lambda^s\Theta/16)|^2}e^{-\lambda^s\pi\Theta/16}. \tag{57}$$

Let us notice that we have neglected the factor $(1-\lambda^s kf'/a^2)^{-1/2}$ because kf'/a^2 is proportional to $k\eta$ and hence negligible on large scales. The above expression is similar to Eq. (15) of Ref. [10]. At this stage, the only thing which remains to be done is to expand the above expression at first order in the slow-roll parameter. After lengthy but straightforward calculations, one obtains the following result:

$$k^3P_h^s(k) = \frac{16H_{\rm inf}^2}{\pi m_{\rm Pl}^2}\frac{1}{2}\mathcal{A}^s(\Theta)\times\left[1-2(C+1)\epsilon-2\epsilon\ln\frac{k}{k_*}-\epsilon\mathcal{B}(\Theta)\right], \tag{58}$$

with

$$\mathcal{A}^{s}(\Theta) \equiv \frac{16}{\pi\Theta}\left(1+\frac{\Theta^2}{256}\right)^{-1}\sinh\left(\frac{\pi\Theta}{16}\right)\exp\left(-\lambda^{s}\frac{\pi\Theta}{16}\right), \tag{59}$$

$$\mathcal{B}(\Theta) \equiv \Psi\left(2-i\frac{\Theta}{16}\right)+\Psi\left(2+i\frac{\Theta}{16}\right)-2\Psi(2). \tag{60}$$

At this point some remarks are in order. As required one can check that, when $\Theta = 0$, the standard inflationary result is recovered. This is the case because $\mathcal{A}^s(0) = 1$ and $\mathcal{B}(0) = 0$. As already mentioned, a factor $1/2$ is left because the (now identical) contribution from the two states of polarization should be added. The function $\mathcal{A}^s$ describes the dominant modification in the amplitude of the power spectrum (the contribution originating from the function $\mathcal{B}$ is clearly subdominant since it is proportional to the slow-roll parameter ϵ). For small values of Θ we have

$$\mathcal{A}^{\mathrm{R}} = 1-\frac{\pi}{16}\Theta+\left(\frac{\pi^2}{384}-\frac{1}{256}\right)\Theta^2+\mathcal{O}(\Theta^3), \tag{61}$$

$$\mathcal{A}^{\mathrm{L}} = 1+\frac{\pi}{16}\Theta+\left(\frac{\pi^2}{384}-\frac{1}{256}\right)\Theta^2+\mathcal{O}(\Theta^3), \tag{62}$$

where $\pi/16 \simeq 0.2$ and $(\pi^2/384 - 1/256) \simeq 0.022$. Therefore, the amplitude of the right polarization state is reduced while the one of the left polarization state is enhanced. However, for small values of Θ, the effect is clearly not very important.

Another conclusion that can be obtained from the above spectrum is that, at leading order in the slow-roll parameter, the spectral index remains unmodified. Indeed, one has $n_{\mathrm{T}}^{s} = d\ln(k^3P_h^s)/d\ln k = -2\epsilon$ for each polarization state.

Finally, let us now compute how the ratio T/S is modified. The scalar power spectrum is not modified (see also Ref. [4]) and reads [9]

$$k^3P_\zeta = \frac{H_{\mathrm{inf}}^2}{\pi m_{\mathrm{Pl}}^2\epsilon}\times\left[1-2\epsilon-2C(2\epsilon-\delta)-2(2\epsilon-\delta)\ln\frac{k}{k_*}\right]. \tag{63}$$

Therefore, we conclude that the consistency check of inflation, at leading order in the slow-roll parameters, can now be written as

$$\frac{T}{S} \equiv \frac{1}{(k^3P_\zeta)}\left(\sum_{s=\mathrm{L,R}}k^3P_h^s\right)\Bigg|_{k=k_*} \tag{64}$$

$$= 16\epsilon\frac{1}{2}[\mathcal{A}^{\mathrm{L}}(\Theta)+\mathcal{A}^{\mathrm{R}}(\Theta)] \tag{65}$$

$$\simeq 16\epsilon\left[1+\left(\frac{\pi^2}{384}-\frac{1}{256}\right)\Theta^2\right]. \tag{66}$$

Unfortunately, the linear corrections in Θ cancel out and we are left with a correction which is quadratic in Θ. Another way to express the above result is to calculate the ratio of T/S with the Chern-Simons modification taken into account to T/S obtained in the standard case. One gets

$$\frac{(T/S)_{\Theta\neq0}}{(T/S)_{\Theta=0}} \simeq 1+0.022\times\Theta^2. \tag{67}$$

It is clear from this expression that the modification is not observable at all since we have seen before that, typically, $\Theta \lesssim 10^{-5}$ in order for the calculations presented here to be consistent (i.e. for the divergence of the effective potential to be in the trans-Planckian region).

IV. DISCUSSION AND CONCLUSIONS

We have evaluated the super-Hubble power spectrum and the tensor to scalar ratio for birefringent gravitational waves produced during inflation. The power spectrum exhibits two interesting regimes, linear and nonlinear. The nonlinear regime occurs when $k\eta \sim \Theta^{-1}$ because the effective potential controlling the evolution of the linear perturbations blows up. At this point, the linear theory of cosmological perturbations that we used is no longer valid. This divergence occurs for all modes (i.e. for all comoving wave number k) but at different times.

In this present investigation we only considered the linear regime since at the present moment we were not able to perform a rigorous analysis of the nonlinear phenomena. We found corrections which survive to second order in Θ. Therefore, in this regime the tensor to scalar ratio gets corrected by Θ but this effect is very small.

If $\Theta \lesssim 10^{-5}$, one can push the nonlinear regime (i.e. the divergence in the effective potential) into the trans-Planckian region where, anyway, other effects (for instance, nonperturbative stringy effects) are likely to become important. Somehow, this corresponds to the standard situation where the evolution from the Planck scale to the superhorizon scales is under control and where the perturbations are assumed to emerge from the trans-Planckian regime in the vacuum state, thus ignoring the modifications of the initial conditions that the trans-Planckian physics could cause (in the very same way that we have ignored the effect of the divergence in the potential, provided it is in the trans-Planckian region). However, it is important to keep in mind that this is mostly a technical trick which allows us to work with the linear theory. At a deeper level, the trans-Planckian effects are not expected to play a more important role than in the standard situation. In particular, if the divergence is not in the trans-Planckian regime, only the nonlinear theory of cosmological perturbations is necessary in order to calculate the modified T/S irrespectively of any trans-Planckian effects.

It is interesting to note that the linear regime (where $\Theta \sim 10^{-5}$) is compatible with the stringy embedding of inflationary baryogenesis [2]. In this context, the value of Θ enhances and gives the resonant frequency associated with the observed baryon asymmetry. As already mentioned before, this value is completely fixed by the string scale and coupling in a model independent fashion. Explicitly, the value of the number $\mathcal{N}$ which appears in the definition of Θ, see Eq. (31), is given by

$$\mathcal{N} = \pi^2\sqrt{\frac{g_s}{2}}\left(\frac{M_{\rm Pl}}{M_{10}}\right)^2, \tag{68}$$

where M_{10} is the ten-dimensional fundamental scale and g_s is the string coupling. Therefore, we established a direct link between stringy quantities and CMB anisotropies. Explicitly, Eq. (67) can be rewritten as

$$\frac{(T/S)_{\Theta\neq 0}}{(T/S)_{\Theta=0}} \simeq 1 + \frac{0.022}{4}\left(\frac{H_{\rm inf}}{M_{10}}\right)^4 g_s\epsilon. \tag{69}$$

In a recent paper the authors of Ref. [2] found that for reasonable values of string coupling (i.e. weak) and the string scale set to 10^{16} GeV, both Θ can be as small as 10^{-2} and the observed baryon asymmetry can be generated. Of course the stringy embedding admits much larger values of Θ putting our analysis into the nonlinear regime.

If $\Theta \gtrsim 10^{-5}$, a nonlinear calculation is mandatory and one can hope to obtain a significative modification of the ratio T/S, maybe observable by future high accuracy CMB experiments. At this point, it is worth mentioning that the birefringence of the gravitational waves could manifest itself in other CMB observables. As was pointed for the first time in Ref. [3], parity violation can induce nonvanishing multipole moments $C_\ell^{\rm TB} = \langle a_{\ell m}^{\rm T}(a_{\ell m}^{\rm B})^*\rangle$, where $a_{\ell m}^{\rm T}$ and $a_{\ell m}^{\rm B}$ are the coefficients of temperature anisotropy and B polarization (also known as curl polarization) in a spherical harmonics expansion. Usually, these multipoles are zero because there is no source of parity violation. Given the order of magnitudes presented before, there is no hope to detect this effect in the linear regime. However, if the parameters of the underlying model are such that the linear approximation is no longer valid, trying to detect a nonvanishing $C_\ell^{\rm TB}$ is certainly another way to probe the presence of birefringent gravitational waves in the CMB. Clearly, the present situation is not very satisfactory since the regime for which sizable effects are expected turns out to be very complicated from the technical point of view. Furthermore, we suspect that the scale associated to the divergence of the effective potential corresponds to a resonant production of lepton number. We wish to report on this issue in a future paper.

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[1] S. Alexander, M. Peskin, and M. Sheikh-Jabbari, hep-th/0403069.
[2] S. H. S. Alexander and S. J. J. Gates, hep-th/0409014.
[3] A. Lue, L. M. Wang, and M. Kamionkowski, Phys. Rev. Lett. **83**, 1506 (1999).
[4] K. Choi, J. Hwang, and K. Hwang, Phys. Rev. D **61**, 084026 (2000).
[5] L. Pogosian, T. Vachaspati, and S. Winitzki, New Astron. Rev. **47**, 859 (2003).
[6] R. Jackiw and S.-Y. Pi, Phys. Rev. D **68**, 104012 (2003).
[7] K. R. S. Balaji, R. H. Brandenberger, and D. A. Easson, J. Cosmol. Astropart. Phys. 12 (2003) 008.
[8] D. H. Lyth, C. Quimbay, and Y. Rodriguez, J. High Energy Phys. 03 (2005) 016.
[9] J. Martin and D. J. Schwarz, Phys. Rev. D **62**, 103520 (2000); Astrophys. J. **543**, L99 (2000).
[10] J. Martin and D. J. Schwarz, Phys. Lett. B **500**, 1 (2001).
[11] I. S. Gradshteyn and I. M. Ryshik, *Table of Integrals, Series and Products* (Academic, New York, 1980).
[12] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1964).
[13] J. Martin and R. H. Brandenberger, Phys. Rev. D **63**, 123501 (2001); R. H. Brandenberger and J. Martin, Mod. Phys. Lett. A **16**, 999 (2001).
[14] M. Lemoine, M. Lubo, J. Martin, and J. P. Uzan, Phys. Rev. D **65**, 023510 (2002); J. C. Niemeyer, Phys. Rev. D **63**, 123502 (2001); A. Kempf, Phys. Rev. D **63**, 083514 (2001); R. Easther, B. R. Greene, W. H. Kinney, and G. Shiu, Phys. Rev. D **64**, 103502 (2001); J. Martin and R. H. Brandenberger, Phys. Rev. D **68**, 063513 (2003); J. Martin and C. Ringeval, Phys. Rev. D **69**, 083515 (2004); **69**, 127303 (2004); J. Cosmol. Astropart. Phys. 01 (2005) 007.