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Mary Lowe

Jerry P. Gollub
Haverford College

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Mary Lowe and J. P. Gollub
Physics Department, Haverford College, Haverford, Pennsylvania 19041, and Physics Department, University of Pennsylvania, Philadelphia, Pennsylvania 19104
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We present an experimental study of the space and time evolution of the Eckhaus instability, a general mechanism of pattern selection for spatially periodic patterns in nonlinear systems. Using a convecting liquid crystal layer, we observed long-wavelength modulations leading to the nucleation of new roll pairs. The development of this process is studied by time-resolved spatial Fourier analysis and compared with predictions based on an amplitude equation.

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The onset of thermal convection in a thin fluid may be characterized by a parabolic linear stability curve in the parameter space formed by the temperature difference across the layer and the wave number of the growing disturbance. However, the phenomenon is complicated by the fact that *two* stability curves are tangent at the critical wave number k_c: the linear stability curve (below which a small-amplitude periodic flow will decay) and the boundary of the Eckhaus instability. The latter causes a flow pattern whose wave number is too far from the critical value to go unstable through the development of slow spatial modulations. This leads to the nucleation or elimination of roll pairs, and eventually produces a new pattern with a wave number closer to the critical value. Thus, the Eckhaus instability is an important mechanism of pattern selection that can lead to a substantial change in the wave number. This process is quite general, since a similar phenomenon occurs in any translationally invariant system where a normal bifurcation produces a periodic structure in one space direction.[1,2]

The Eckhaus instability is difficult to observe, both because it is often masked by secondary instabilities,[3] and because lateral boundaries limit the range of stable wave numbers.[4] Nevertheless, the stability boundary has been measured in circular Couette flow[5] and buckling beams[6]; limited observations have also been made on hydrodynamic surface waves and on Rayleigh-Bénard convection rolls in an annular container.[7] In this paper, we present the first direct observations of the space and time evolution of the Eckhaus instability. In earlier experimental studies[5,6] the Eckhaus boundary was approached gradually from the stable side. In that case, the instability occurs at zero wave number so that the spatial modulations observed in the present work could not be seen.

We use electrohydrodynamic (rather than thermal) convection to obtain a sample containing at least 150 rolls. This is sufficiently large that finite-size effects should be irrelevant on the time scale of the experiment. By controlling the initial wave number of the roll pattern and the layer depth, we are able to make precise measurements of the stability boundaries and the time evolution of various spatial Fourier components of the pattern. The stability curve and the wave number of the secondary flow are compared with predictions based on an amplitude equation, an expansion in powers and derivatives of a slowly varying field.

The working fluid is a nematic liquid crystal [N-*p*-methoxybenzylidine-*p*-butylaniline (MBBA)] confined between two transparent conductive electrodes with an adjustable separation $d = 20$–120 μm. A potential difference V larger than V_c (≈ 6 V ac) induces a one-dimensional roll pattern with critical wave number k_c (on the order of $\frac{1}{2}d$) similar to that resulting from the Rayleigh-Bénard instability. We define a control parameter $\epsilon \equiv (V - V_c)/V_c$ and a dimensionless wave number $Q \equiv (k - k_c)/k_c$. The nematic director is aligned in a particular horizontal direction by surface treatment, and the rolls are always perpendicular to this direction. This alignment has the effect of suppressing two-dimensional instabilities that might compete with the Eckhaus mode.

The experiment was performed by creating a roll pattern with a fixed wavelength of 200 μm, using a small spatially periodic field produced by an interdigitized electrode.[8] The periodic field is then eliminated without a change in ϵ, and the pattern is allowed to evolve to a steady state that depends on the (variable) layer thickness. If the initial conditions are located in the Eckhaus unstable region of the phase diagram (see Fig. 2), the initially periodic pattern develops a long-wavelength modulation, as shown in Fig. 1. [See also the light intensity measurements in Fig. 3(a).] The pattern is visible because the convection is associated with a periodic tilt of the director out of the horizontal plane. This produces a spatially varying index of refraction that focuses transmitted light.

We have measured the Eckhaus stability boundary $\epsilon_E(Q)$ for $Q < 0$, as shown in Fig. 2. The Eckhaus modulations occur only to the left of $\epsilon_E(Q)$. If the initial conditions are chosen to lie to the right of $\epsilon_E(Q)$ (closer to $Q = 0$), the wave number still evolves to-

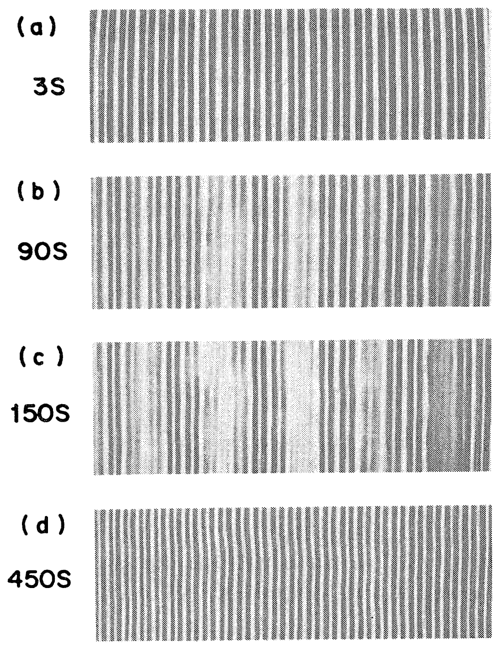

FIG. 1. Development of the Eckhaus instability from a spatially periodic pattern for $\epsilon = 0.052$. New roll pairs nucleate in the regions of weak optical contrast, leading to a higher wave number. Only a *small part* of the sample is shown. In the initial pattern, $|Q| = 0.194$. Refer to Fig. 3(a) for a graph of the optical intensity vs position.

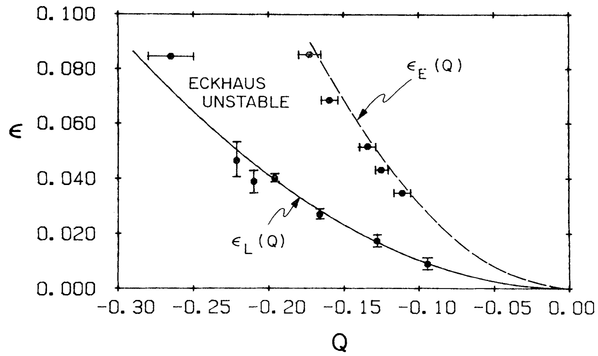

FIG. 2. Measured stability boundaries as a function of the distance ϵ above onset and the dimensionless wave number Q. The solid line is a parabolic fit to the data for the linear stability curve $\epsilon_L(Q)$, while the dashed line is the prediction for the Eckhaus boundary $\epsilon_E(Q) = 3\epsilon_L(Q)$.

ward $Q = 0$, but by a different process involving dislocation motion. This latter process is slower by at least an order of magnitude.

We also found the *linear* stability boundary $\epsilon_L(Q)$, where the decay time of a small-amplitude periodic flow diverges when approached from below. To measure $\epsilon_L(Q)$, a small-amplitude periodic state is prepared by external forcing at $\epsilon < 0$. (By our keeping ϵ negative, the growth of other modes is suppressed.) We then remove the forcing and increase ϵ abruptly to $0 < \epsilon < \epsilon_L(Q)$. The decay rate of the convective amplitude vanishes linearly with ϵ at $\epsilon_L(Q)$, thus allowing a determination of this boundary by extrapolation to zero decay rate. The solid line in Fig. 2 is a parabolic fit to the experimental data for the linear stability curve. [Although a precise comparison is not possible on the basis of published computations, $\epsilon_L(Q)$ is reasonably consistent with numerical results for MBBA.[9]]

When ϵ is small, an amplitude equation is believed to provide a good description of the Eckhaus instability.[2] Within this approximation, $\epsilon_L(Q)$ is parabolic, and $\epsilon_E(Q) = 3\epsilon_L(Q)$. The dashed line in Fig. 2 shows this prediction for $\epsilon_E(Q)$, assuming $\epsilon_L(Q)$ is determined experimentally. The experimental data agree satisfactorily with it.

In order to study the evolution of the pattern selection process that results from the Eckhaus instability, we utilized time-resolved spatial Fourier analysis of the digitized optical intensity patterns. After the Eckhaus instability develops, the intensity function $I(x)$ has clear modulations, as shown in Fig. 3(a), in which each rapid oscillation corresponds to one roll. The intensity is related (to lowest order) to the square of the tilt angle of the director field. The spectrum therefore exhibits a peak at $2k_1$, where k_1 is the initial wave

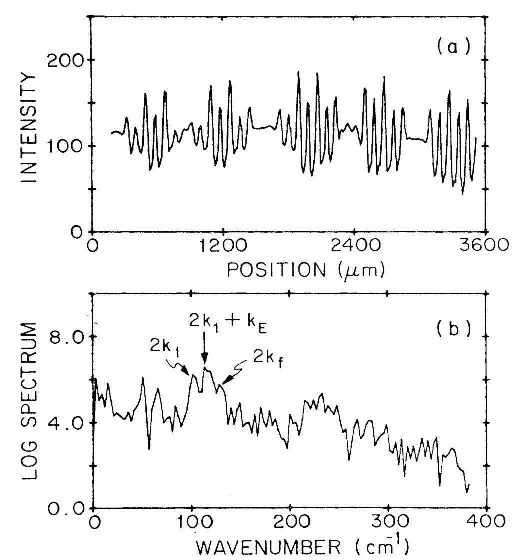

FIG. 3. (a) Optical intensity as a function of position, transverse to the roll axis ($t = 180$ s). The Eckhaus instability is evident as a pronounced periodic spatial modulation. ($\epsilon = 0.052$ and $|Q| = 0.194$ initially.) (b) Corresponding power spectrum, showing the initial peak at $2k_1$, and the Eckhaus sidebands at $2k_1 + k_E$ and $2k_1 + 2k_E (= 2k_f)$.

number of the pattern. Amplitude and phase modulation with wave number k_E lead in principle to sum and difference wave numbers so that there should be spectral components at $2k_1 \pm k_E$ and $2k_1 \pm 2k_E$. In fact, we usually observe only the positive sidebands. The peak at $2k_1 - k_E$ appears only for initial patterns located extremely close to the Eckhaus boundary. This is consistent with theory,[10] which predicts a Q-dependent asymmetry in the power of the sidebands. [A small peak at k_E is also visible in Fig. 3(b), as well as unlabeled peaks near 50 cm^{-1} corresponding to roll pairs.]

The wave number k_E of the Eckhaus modulation (in units of k_c) is plotted as a function of Q in Fig. 4(a) for fixed $\epsilon = 0.052$. The data indicate a trend toward decreasing k_E as Q approaches the Eckhaus boundary $Q_E(\epsilon)$, located at $Q_E = 0.13$ for $\epsilon = 0.052$. [$Q_E(\epsilon)$ is the inverse of $\epsilon_E(Q)$, shown in Fig. 2.] The solid line represents the theoretical prediction[10] for the *most rapidly growing* mode: $(k_E/k_c)^2 = Q^2 - [3Q_E^2 - Q^2]^2/4Q^2$. Actually, the system is unstable with respect to a *band* of wave numbers $0 < (k_E/k_c)^2 < 6(Q^2 - Q_E^2)$. The dashed line represents the upper bound. Although the scatter is substantial (and greater than the measurement error), the experimental data lie above the most rapidly growing mode but within the unstable band.

Although the various spectral peaks are close together, it is possible to follow their intensities A_k, defined to be the areas under the corresponding spectral peaks, as functions of time. An example is shown in Fig. 4(b). The initial peak at $2k_1$ decays partly because of the large initial flow amplitude (created by periodic forcing), and partly because of the growth of the Eckhaus peak at $2k_1 + k_E$. The latter initially grows exponentially at a rate roughly proportional to the distance (measured horizontally in Fig. 2) from the Eckhaus stability boundary. It subsequently decays as the peak at $2k_f(= 2k_1 + 2k_E)$ becomes comparable in size. The latter eventually saturates and becomes the dominant peak in the spectrum of the final stable pattern. It is a result of the nucleation of a roll pair at each wavelength $1/k_E$ of the modulation. We find that k_f is surprisingly close to band center: $|k_f - k_c|/ k_c < 0.06$.

The time development of the nucleation of a roll pair is shown in detail by the optical intensity $I(x)$ in Fig. 5. Each full oscillation corresponds to a single roll. In this example, the flow amplitude modulates to zero near 1500 μm at $t = 180$ s after the initial periodic pattern is established. Then two new rolls appear (lowest panel in Fig. 5), thus increasing the wave number of the pattern. We presume, but have not checked, that roll pairs would be eliminated (rather than nucleated) if $Q < 0$. The late stages of the evolution are sensitive to the phase of the rolls, and therefore cannot be described by an amplitude equation that ignores fast variations. We note that Kramer and Zimmerman[2] have performed numerical simulations of the roll nucleation process based on the Swift-Hohenberg

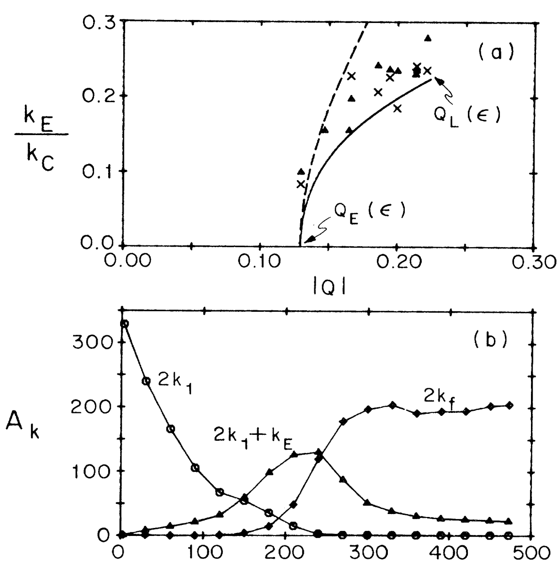

FIG. 4. (a) Wave number k_E of the Eckhaus modulation (in units of k_c) from real-space (triangles) and spectral (crosses) measurements as a function of the dimensionless initial wave number Q, for $\epsilon = 0.052$. It vanishes at the Eckhaus stability boundary Q_E. Q_L is the wave number of the linear stability curve at $\epsilon = 0.052$. The solid line is the theoretical prediction for the most rapidly growing wave number, whereas the dashed line represents the upper bound of the unstable band. (b) Time evolution of the power A_k of the spectral peaks at $2k_1$, $2k_1 + k_E$, and $2k_f$. (Here, $|Q| = 0.194$ for the initial pattern.)

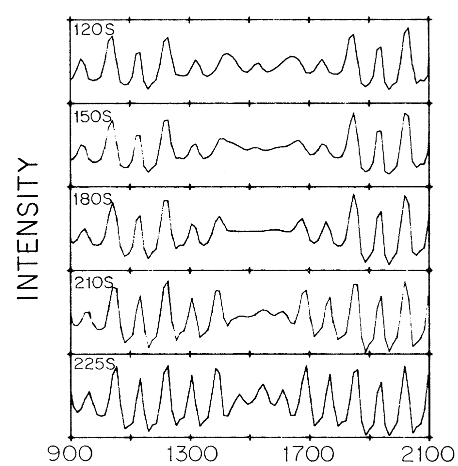

FIG. 5. Detail of the optical intensity $I(x)$ at various times for fixed $\epsilon = 0.052$. The nucleation of a new roll pair (two extra full oscillations) at a minimum of the *envelope* of $I(x)$ is visible in the lower two panels. (Initially, $|Q| = 0.194$.)

model.

Before concluding, we note several nonideal features of this process. First, the Eckhaus modulations are not perfectly coherent: The spacing between the modulations fluctuates somewhat. Second, the initial forcing could possibly affect the wave number of the subsequent modulations to a small extent. Third, there are cases in which the pattern evolution does not remain completely one dimensional. When the initial conditions are close to $\epsilon_E(Q)$, the Eckhaus instability occurs along segments of the rolls. This process leaves diffuse dislocations that climb rapidly along the roll axis. Thus the Eckhaus instability leads smoothly into dislocation motion, which dominates the process of pattern selection for $|Q| < |Q_E|$.

We have presented a quantitative study of the space and time evolution of the Eckhaus instability, a general mechanism of pattern selection for periodic structures. We find that the Eckhaus stability boundary is in good agreement (for $\epsilon < 0.1$) with predictions based on an amplitude equation. The wave number $k_E(Q)$ of the modulations lies within the expected band but is higher than the most rapidly growing wave number obtained from linear theory. As a result, the Eckhaus instability is somewhat more effective in driving the wave number of the pattern toward k_c than the linear theory would suggest. The question of whether the finite initial forcing can significantly affect the instability is being addressed theoretically.[10] The nonlinear stages of the evolution, especially the nucleation and growth of new rolls, are not yet quantitatively understood. This experiment demonstrates that the linear stability curve itself does not delineate the allowed periodicities of stable convective flows.

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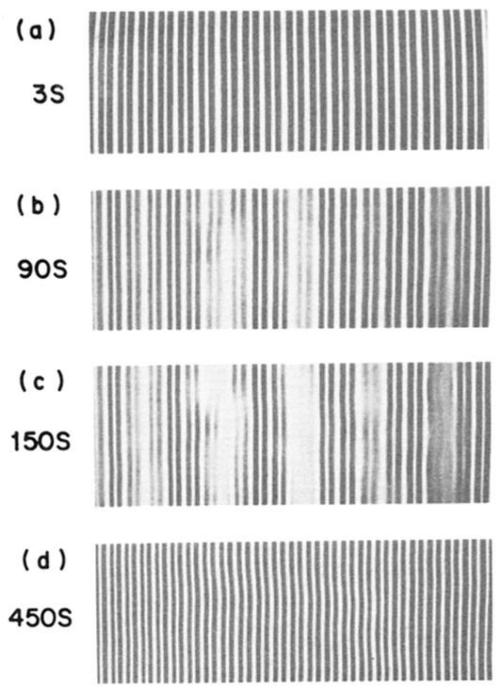

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