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Summary. — We present a four-variable model (two coupled parametrically forced oscillators) that describes many of the phenomena seen in an experiment on surface waves in which the competition between spatial patterns produces chaotic behaviour. The model reproduces the route to chaos, the dimension of the attractor, the Kolmogorov entropy and (approximately) the phase diagram.

PACS. 47.25. – Turbulent flows, convection and heat transfer.
PACS. 47.35. – Hydrodynamic waves.

1. – Introduction.

In a recent paper [1] we described an experiment on forced surface waves in which chaotic behaviour is clearly produced by the competition between two spatial modes. We also suggested a phenomenological model that explains many of the experimental results. In this report we describe the phenomenological model in greater detail. The experiment itself is thoroughly treated in a separate publication [2].

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(1) S. CILIBERTO and J. P. GOLLUB: *Phys. Rev. Lett.*, **52**, 922 (1984).
(2) S. CILIBERTO and J. P. GOLLUB: *J. Fluid Mech*, **158**, 381 (1985).

We recall only the results of the experiment to allow a direct comparison with those of the model. The system of interest is a cylindrical fluid layer in a container that is subjected to a small vertical oscillation. It is well known ([3]) that, if the driving amplitude exceeds a critical value $A_c(f_0)$, which is a function of frequency, the free surface develops a pattern of standing waves. The surface deformation $S(r, \theta, t)$ can then be written as a superposition of normal modes:

$$S(r, \theta, t) = \sum_{l,m} a_{l,m}(t) J_l(k_{l,m} r) \cos l\theta \,, \tag{1}$$

where J_l are Bessel functions of order l and the allowed wave numbers $k_{l,m}$ are determined by the boundary condition that the derivative $J'_l(k_{l,m}R) = 0$, where R is the radius of the cylinder. The modes may be labeled by the indices l (giving the number of angular maxima) and m (related to the number of nodal circles). The mode amplitude $a_{l,m}(t)$ develops an instability when the corresponding eigenfrequency (given by the dispersion law for capillary gravity waves) is approximately in resonance with half the driving frequency f_0 and A exceeds $A_c(f_0)$. This parametric instability leads to standing waves in which the mode amplitude oscillates at $f_0/2$. To take into account the possibility of a further slow modulation of the mode amplitudes, which, in fact, occurs due to mode competition, we write each amplitude in terms of fast oscillations at $f_0/2$ and slow envelopes $C_l(t)$ and $B_l(t)$:

$$a_l(t) = C_l(t) \cos(\pi f_0 t) + B_l(t) \sin(\pi f_0 t) \,. \tag{2}$$

We omit the second subscript because, in practice, only a single value of m is significant for a given value of l.

The behavior of the system as a function of A and f_0 is shown in fig. 1. Below the parabolic stability boundaries, the surface is essentially flat. Above the stability boundaries, the fluid surface oscillates at half the driving frequency in a single stable mode. The shaded areas are regions of mode competition, in which the surface can be described as a superposition of the (4, 3) and (7, 2) modes with amplitudes having a slowly varying envelope in addition to the fast oscillation at $f_0/2$.

Our experimental apparatus, described in ([1,2]), allows us to study a fixed linear combination of the slow coefficients $C_l(t)$ and $B_l(t)$, which we denote by $a_l^0(t)$. The dynamics of the slow oscillation was explored by varying A and f_0 separately inside of the interaction region. In fig. 2 time series and corresponding power spectra are shown for three different driving amplitudes but fixed driving frequency of 16.05 Hz.

([3]) T. B. Benjamin and F. Ursell: *Proc. R. Soc. London, Ser. A*, **225**, 505 (1954).

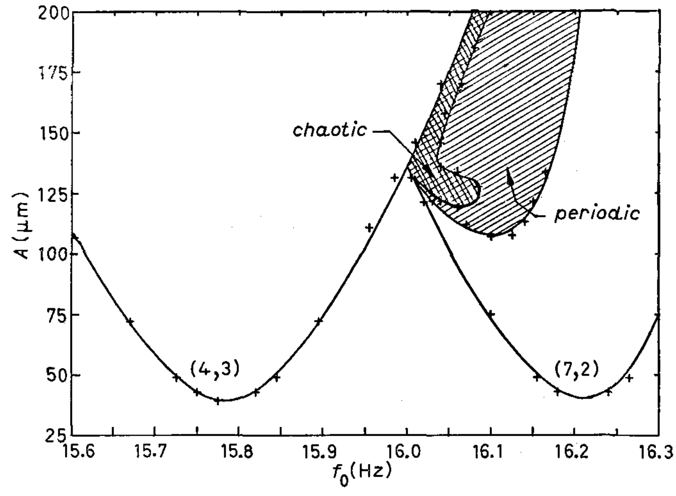

Fig. 1. – Phase diagram as a function of driving amplitude A and frequency f_0. The crosses are experimentally determined points on the stability boundaries. Stable patterns occur in the regions labeled (4, 3) and (7, 2). Slow periodic and chaotic oscillations involving *competition* between these modes occur in the shaded regions.

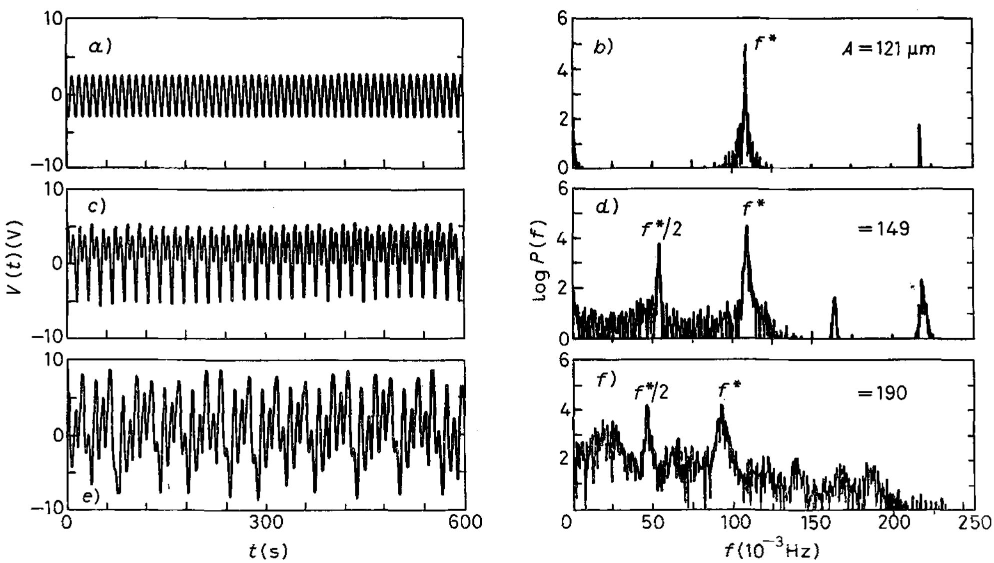

Fig. 2. – The transition from periodic to chaotic oscillation. Time series and corresponding power spectra of the slow oscillation are shown for $f_0 = 16.05$ Hz and three different driving amplitudes. Broad-band noise is associated with the appearance of a subharmonic $f^*/2$ of the dominant oscillation.

As the driving amplitude is increased, a chaotic state with a broad power spectrum is obtained. We characterized the chaotic behaviour quantitatively [2] by computing from the experimental data the correlation dimension ν of the attractor [4] and a lower bound K_2 for the Kolmogorov entropy K [5]. When the oscillation is periodic ($A = 121\ \mu\text{m}$), we find $\nu = 1.0 + 0.04$ and $K_2 = (0.0 \pm 0.01)\ \text{s}^{-1}$. On the other hand, when the slow oscillation is chaotic ($A = 190\ \mu\text{m}$), $\nu = 2.22 \pm 0.04$ and $K_2 = (0.1 \pm 0.01)\ \text{s}^{-1}$. These measurements clearly demonstrate that the attractor has a low (and fractional) dimension and that there is at least one positive Lyapunov exponent. The dimension measurements also show that a four-dimensional phase space is required to represent the data.

2. – Formulation of the model.

We have constructed a relatively simple phenomenological model that has a reasonable hydrodynamic basis and accounts for most of our observations, including the basic structure of the phase diagram. We begin with the fact that, in a linearized inviscid approximation, each mode amplitude $a_l(t)$ follows a Mathieu equation [3]:

$$\ddot{a}_l(t) + (\omega_l^2 - \psi_l A \cos \omega_0 t)\, a_l(t) = 0\,, \tag{3}$$

where ω_l is the eigenfrequency, ψ_l is a gain coefficient and $\omega_0 = 2\pi f_0$. We take the point of view that one can approximately account for the effects of damping (due to all sources, including bulk viscosity and wall effects) by introducing a first-order term $\gamma_l \dot{a}_l$. Furthermore, it is necessary to add a nonlinear term to limit the growth of the mode to finite amplitude in the steady state. The lowest-order nonlinear term is cubic in the mode amplitude, thus we have the following equation for the time variation of any mode:

$$\ddot{a}_l + \gamma_l \dot{a}_l + (\omega_l^2 - \psi_l A \cos \omega_0 t)\, a_l = \zeta_l a_l^3\,. \tag{4}$$

We find that this equation is sufficient to fit the (approximately parabolic) stability curves in fig. 1 and to describe quantitatively the variation of the steady-state mode amplitude with A above threshold.

Next we consider the phenomenon of mode competition. We know that only two modes are involved and that they interact. Therefore, we consider a model consisting of two coupled Mathieu oscillators. There are various ways to introduce coupling phenomenologically. One may allow the coefficient of

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(4) P. GRASSBERGER and I. PROCACCIA: *Physica (Utrecht) D*, **9**, 189 (1983).

(5) P. GRASSBERGER and I. PROCACCIA: *Phys. Rev. A*, **28**, 2591 (1983).

the driving term for one mode to depend on the amplitude of the other mode. Alternatively, one may allow each damping coefficient to depend on the amplitude of the other mode. Finally, one may introduce nonlinear terms containing both mode amplitudes. We have tried all three approaches, but we present results only for the first one, simply because it provides a better fit to the experiments. Therefore, to describe the interaction of the $l = 7$ and $l = 4$ modes, we set

$$\psi_4 = \bar{\psi}_4 + \beta_{47} a_7^2 \quad \text{and} \quad \psi_7 = \bar{\psi}_7 + \beta_{74} a_4^2, \tag{5}$$

where the coupling coefficient β_{74} is negative, while β_{47} is positive. The origin of the sign difference is the observed phase difference between the two modes during the oscillation. In fact, a_7^0 leads a_4^0 by about 90° (see fig. 4 of ref. [1]). This implies that the sevenfold mode pumps the fourfold mode, while the fourfold mode damps the sevenfold mode. To solve the system of the two coupled Mathieu equations, we express the mode amplitudes by eq. (2). We substitute eqs. (2) and (5) into (4), keeping only terms oscillating at πf_0 and neglecting those at $3\pi f_0$. We also neglect $\ddot{B}_l$ and $\ddot{C}_l$ because the time scales of the fast oscillation and mode competition are very different, so that $\ddot{B}_l \ll 2\omega_0 \dot{C}_l$ and $\ddot{C}_l \ll 2\omega_0 \dot{B}_l$. Finally, we obtain the following four-dimensional system for the slow variables C_4, B_4, C_7 and B_7:

$$\begin{cases} \dot{C}_4 = -\frac{1}{2}\gamma_4 C_4 - [\Delta_4 - \psi_4^0 A + \zeta_4^0(C_4^2 + B_4^2) - \beta_{47}^0 B_7^2] B_4, \\ \dot{B}_4 = -\frac{1}{2}\gamma_4 B_4 + [\Delta_4 + \psi_4^0 A + \zeta_4^0(C_4^2 + B_4^2) + \beta_{47}^0 C_7^2] C_4, \\ \dot{C}_7 = -\frac{1}{2}\gamma_7 C_7 - [\Delta_7 - \psi_7^0 A + \zeta_7^0(C_7^2 + B_7^2) - \beta_{74}^0 B_4^2] B_7, \\ \dot{B}_7 = -\frac{1}{2}\gamma_7 B_7 + [\Delta_7 + \psi_7^0 A + \zeta_7(C_7^2 + B_7^2) + \beta_{74}^0 C_4^2] C_7, \end{cases} \tag{6}$$

where we have introduced normalized coefficients as follows:

$$\begin{cases} \Delta_l = (\omega_0^2 - 4\omega_l^2)/4\omega_0, & \psi_l^0 = \bar{\psi}_l/\omega_0, \\ \beta_{47}^0 = \beta_{47} A/2\omega_0, & \beta_{74}^0 = \beta_{74} A/2\omega_0, \\ \zeta_l^0 = 3\zeta_l/4\omega_0. & \end{cases} \tag{7}$$

3. – Integration and properties of the model.

In this system most of the coefficients can be *measured*. The damping coefficients γ_l and gain coefficients ψ_l^0 are adjusted to fit the parabolic stability curves of fig. 1, and the nonlinear coefficients ζ_l are chosen to reproduce the measured saturation amplitudes, all in a region where only a single mode is present. We numerically integrate the system using the fourth-order Runge-Kutta method and find that regenerative oscillations (both periodic and chaotic) are, in fact, produced near the intersection of the stability boundaries for the two modes. We adjust the two mode-coupling coefficients to obtain an oscillatory domain similar in size to that found in the experiments (fig. 1). The space diagram produced by this set of model equations is shown in fig. 3. The value of the parameters used in the model are

$$\gamma_4 = \gamma_7 = 0.40\ \text{s}^{-1}, \quad \psi_4^0 = 51.3\ \text{cm}^{-1}\,\text{s}^{-1}, \quad \psi_7^0 = 52.6\ \text{cm}^{-1}\,\text{s}^{-1}, \quad \zeta_4^0 = 1.00\ \text{s}^{-1},$$
$$\zeta_7^0 = 0.10\ \text{s}^{-1}, \quad \beta_{47}^0 = +(7.0\ \text{cm}^{-1}\,\text{s}^{-1})A, \quad \beta_{74}^0 = -(9.0\ \text{cm}^{-1}\,\text{s}^{-1})A,$$
$$\omega_4 = 49.59\ \text{rad/s}, \quad \omega_7 = 50.92\ \text{rad/s}.$$

(The mode amplitudes are taken to be dimensionless with a scale set by the arbitrary chioce of ζ_4^0.)

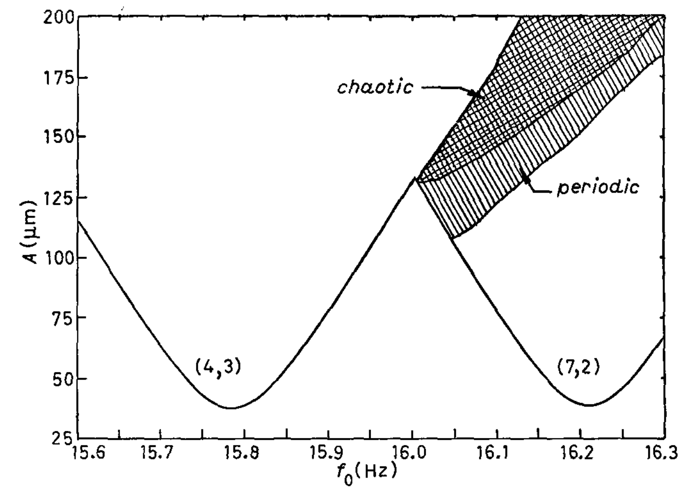

Fig. 3. – Phase diagram obtained from the phenomenological model by numerical integration. This figure should be compared with fig. 1.

The parabolic stability curves fit the experimental data to within about 10 % for $A \leqslant 150\ \mu\text{m}$. However, the shapes of the periodic and chaotic regims are different from those found experimentally.

In order to compare the behaviour near the onset of chaos with that observed experimentally, we present (fig. 4) time series of the slow component B and corresponding power spectra for three different values of A, but fixed $f_0 = 16.11$ Hz. This figure may be compared with the experimental data in fig. 2. The basic period of oscillation is different by a factor of two, an unexplained discrepancy. However, in both cases we find the following features: a single subharmonic bifurcation of the slowly varying mode amplitude for comparable A and f_0; an increase in the background noise level at or near this bifurcation; and a loss of all sharp spectral structure at higher A without further bifurcations. Thus the onset of chaos seems to be quite similar in the data and in the model.

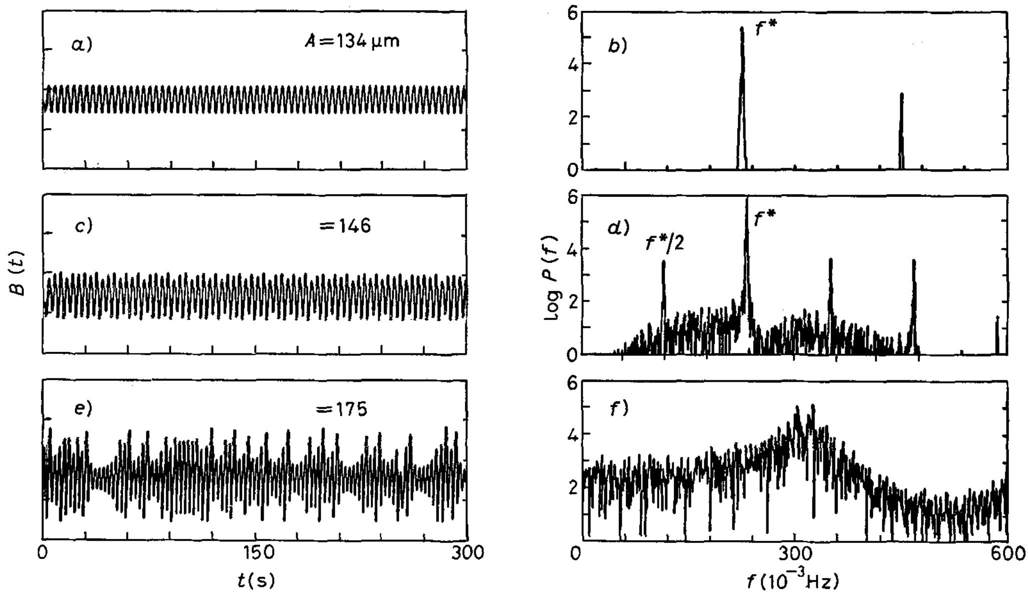

Fig. 4. – Time series and Fourier spectra obtained from numerical integration of the model at three driving amplitudes. This figure illustrates the transition from periodic to chaotic oscillation and should be compared to the experimental data of fig. 2. Both experiment and model show a single subharmonic bifurcation with associated broad-band noise onset.

We also measured the correlation dimension ν for the chaotic states of the model. We find $\nu = 2.41 \pm 0.04$ for $A = 175\ \mu m$, and the same result at $A = 155\ \mu m$. Thus the strange attractor produced by the model has about the same dimension as that found in the experiment.

The Lyapunov exponents of the model equations were also computed with the method proposed in ([6]). The Kolmogorov entropy K (the sum of positive Lyapunov exponents) is $0.33\ s^{-1}$ at $A = 175\ \mu m$. The ratio K/f of the Kolmogorov entropy to the frequency f of the slow oscillation is approximately 1.1 for both the model and the experiment.

([6]) G. BENETTIN, L. GALGANI and J. M. STRELCYN: *Meccanica*, **15**, 9 (1980).

4. – Conclusion.

We have presented a simple phenomenological model for chaotic-mode competition in an experimental study of parametrically forced surface waves. The model describes many of the experimental results fairly well, including the qualitative structure of the parameter space (fig. 3), the route to chaos (fig. 4), the dimension of the resulting strange attractor and the Kolmogorov entropy. (However, the model is not in complete agreement with the data. The period of oscillation is off by a factor of two, for example, and the shapes of the stability boundaries are different.)

One might expect that a correct model for this type of problem would be derivable from a Hamiltonian when the damping terms are eliminated as in (7). The model described in this paper does not satify this condition because the coupling terms must have opposite signs to fit the data. As an alternative, we constructed a four-variable model starting with the Hamiltonian formulation proposed in (7), and adding damping. However, we could not obtain a good fit to the experimental data using this approach. (It is possible, however, that we were simply unable to find the correct parameters empirically).

In the future, it would be desirable to investigate the relationship between low-dimensional models and the actual hydrodynamic equations, instead of using a phenomenological approach as we have done. The phenomenological model does, however, provide a good summary of the experimental results.

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(7) J. Miles: *J. Fluid Mech*, **146**, 285 (1984)..

● RIASSUNTO

Si presenta un modello a quattro variabili (due oscillatori accoppiati forzati parametricamente) che descrive molti dei fenomeni visti in un esperimento sulle onde di superficie nel quale la competizione fra le configurazioni spaziali produce un comportamento caotico. Il modello riproduce la via al caos, la dimensione dell'attrattore, l'entropia di Kolmogorov e (approssimativamente) il diagramma di fase.

Резюме не получено.