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Complete Set of Representations for Dissipative Chaotic Three-Dimensional Dynamical Systems

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Complete set of representations for dissipative chaotic three-dimensional dynamical systems

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Embeddings are diffeomorphisms between some dynamical phase space and a reconstructed image. Different embeddings may or may not be equivalent under isotopy. We regard embeddings as representations of the dynamical phase space. We determine the topological labels required to distinguish inequivalent representations of three-dimensional dissipative dynamical systems when the embeddings are into $\mathbb{R}^k$, $k=3,4,5,\ldots$. Three representation labels are required for embeddings into $\mathbb{R}^3$, and only one is required in $\mathbb{R}^4$. In $\mathbb{R}^5$ there is a single "universal" representation.

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I. INTRODUCTION

Dynamical systems are generally studied through their embeddings into the Euclidean spaces $\mathbb{R}^k$. This is particularly true for data generated by a chaotic dynamical system. Each embedding is a diffeomorphism relating the original system and its attractor with an *image*. The image can be regarded as a *representation* of the original system and its attractor. Since it is obtained through a diffeomorphism, the representation is *faithful*.

As in the theory of groups, it is useful to study the spectrum of representations available to an n-dimensional dissipative dynamical system when embedded into $\mathbb{R}^k$, $k \geq n$. With such information it is possible to address the following important question that has never been adequately addressed: when we study an embedding of a dynamical system, how much of what we learn depends on the embedding and how much depends on the system alone, i.e., is embedding independent?

This question has already been addressed, and representation theory already worked out, for a special but large and important class of dynamical systems, those of genus-one type in $\mathbb{R}^3$ [1]. In the present paper, we extend this program of representation theory to a larger class: three-dimensional dynamical systems of arbitrary genus. More specifically, we study embeddings of dissipative three-dimensional dynamical systems (flows) that generate strange attractors and satisfy the conditions of the Birman-Williams theorem [2,3]. Our results are summarized in Table II.

We are careful here to distinguish two uses of the term *embedding* in dynamical system theory. The first usage means a diffeomorphism of a manifold onto a subset of another [4]. It is a one-to-one smooth mapping with a smooth inverse. The second usage means a one-to-one mapping of the attractor (usually not a manifold) of a dynamical system onto its image. This is the usual sense in the context of reconstructing dynamical systems [5,6]. An embedding of the first type induces an embedding of the second. In this paper, embedding will always be used in the first sense.

In Sec. II we briefly review the spirit of the representation theory of groups and compare it to the spirit of the representation theory of dynamical systems. The similarities and differences are pointed out. In Sec. III we summarize the results obtained in a previous study of the genus-one case. Many of the results for the genus-g case are simple extensions of the previous results. In Sec. IV we present our results. They are presented by comparison with the genus-one results and the motivations underlying these extensions are described. Detailed proofs are not given in this paper: they are available elsewhere [7]. We summarize our results in Sec. V.

II. IDEA OF REPRESENTATION THEORY

The theory of groups is closely associated with the theory of group representations [8]. A representation of a group is a mapping of each group operation into a matrix that preserves the group operations. The mapping can be either an isomorphism (1:1 or faithful) or a homomorphism (many:1 or unfaithful). Equivalence of representations is by similarity transformation. For computational purposes, it is sometimes simpler to carry out calculations in a faithful matrix representation rather than in the group itself.

A representation theory for dynamical systems is under construction [1,7]. At present it has been implemented only for three-dimensional dissipative dynamical systems that satisfy the conditions of the Birman-Williams theorem. It is similar in spirit but different in details from the more familiar representation theory for groups. The basic idea is that a dynamical system can be mapped onto another dynamical system through a smooth mapping. The image system is a representation of the original dynamical system. The mapping can be a diffeomorphism (1:1 or faithful) or not. We consider only faithful representations. Equivalence of representations is by isotopy—we consider two representations of a dynamical system to be equivalent if they are isotopic. Roughly speaking, two embeddings are isotopic if one can be smoothly deformed into the other through a continuous sequence of embeddings (no tearing or gluing allowed).

As for the representation theory of groups, several questions pose themselves. Given a dynamical system, what is the spectrum of inequivalent representations? What representation labels are required to distinguish among inequivalent representations? Other natural questions have no analog in the theory of group representations: as the embedding dimension increases, some representations that are inequiva-

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lent in the lower-dimensional space become equivalent in the higher-dimensional space. Which ones?

Questions of this type are important when considering the reconstruction of dynamical systems through embedding methods. Different embeddings may provide inequivalent representations. As the dimension of the embedding space increases, representations that were inequivalent in a lower-dimensional embedding become equivalent in higher dimensions. There is a limit to this. Takens [5] showed that an n-dimensional dynamical system can always be embedded in $\mathbb{R}^{2n+1}$ using a differential or time delay mapping based on a generic observable, and Wu [9] showed that all embeddings (representations) of an n-dimensional manifold [10] ($n \geq 2$) are isotopic (equivalent) in $\mathbb{R}^{2n+1}$. As a result, an n-dimensional dynamical system can have many inequivalent representation in n dimensions, and as the dimension k increases, the number of inequivalent embeddings decreases. Finally, for $k \geq 2n+1$ there is only one "universal" representation or embedding. A single universal representation may exist for a smaller value of the embedding dimension k, as in the cases discussed below.

III. REVIEW OF GENUS ONE

The details of the representation theory of three-dimensional dissipative dynamical systems of genus one have already been worked out. We review them briefly here as the extension of our results to three-dimensional dynamical systems of higher genus is closely related to the genus-one results. Such dynamical systems are flows inside a torus $\mathcal{T}=D^2 \times S^1$. The results are obtained by a combination of topological arguments combined with dynamical properties that "dress" the torus with a flow.

Three labels are required to distinguish inequivalent representations in $\mathbb{R}^3$. These are oriented knot type $\mathcal{K}$, parity $\mathbb{Z}_2$, and global torsion $\mathbb{Z}$. The knot type of the representation is the knot type of the centerline or core curve of the torus. This core knot may be obtained by simultaneously shrinking each disk D^2 in $\mathcal{T}=D^2 \times S^1$ to its center, yielding the circle $\{0\} \times S^1$. This circle inherits an orientation determined by the direction of the original flow. This oriented core may be thought of as describing the fundamental flow direction of the dynamical system. This fundamental flow direction may be mapped onto any oriented knot in $\mathbb{R}^3$. We denote by $\mathcal{K}$ the set of all oriented knots. Different knots determine different (inequivalent) representations.

The parity of the representation is its handedness or orientation. The orientation reversing diffeomorphism $(x,y,z) \mapsto (x,y,-z)$ of $\mathbb{R}^3$ changes the handedness of the representation. There are exactly two orientations ($\mathbb{Z}_2=\{\pm 1\}$). Different orientations determine different representations.

Global torsion is more subtle [1,11]. A genus-one system has a global Poincaré section consisting of a disk D transverse to the flow. Imagine cutting open $\mathcal{T}$ along this disk, rotating one side of the cut q turns, then reconnecting. If the number of rotations is an integer, the flow will always match up continuously afterward; however, a little care must be taken to ensure smoothness. The integer $q \in \mathbb{Z}$ is the global torsion. Different global torsions determine different representations.

TABLE I. Representation labels for genus-one systems. For parity $\mathbb{Z}_2=\{\pm 1\}$, while for global torsion $\mathbb{Z}_2=\{0,1\}$. $\mathcal{K}$ denotes the set of oriented knots.

	Obstructions to isotopy		
Representation labels	$\mathbb{R}^3$	$\mathbb{R}^4$	$\mathbb{R}^5$
Global torsion	$\mathbb{Z}$	$\mathbb{Z}_2$	
Parity	$\mathbb{Z}_2$		
Knot type	$\mathcal{K}$		

If an embedding with representation labels $(\mathcal{K},\mathbb{Z}_2,\mathbb{Z})$ in $\mathbb{R}^3$ is mapped into $\mathbb{R}^4$, there are fewer obstructions to isotopy, hence fewer representation labels and fewer distinct representations. The fundamental flow direction now follows a knotted circle in $\mathbb{R}^4$. However, it is well known that knots fall apart in $\mathbb{R}^4$: any two simple closed curves are isotopic. This means that there is only one fundamental flow direction for $\mathcal{T}$ to follow in $\mathbb{R}^4$. Knot type disappears as a representation label [1]. Parity also disappears. Global torsion is more complicated. Representations with different global torsions fall into two distinct classes in $\mathbb{R}^4$: all those with q even are equivalent to each other and all those with q odd are equivalent to each other. The global torsion reduces to an integer mod 2, indicating evenness or oddness [1]. This phenomenon is related to the Dirac belt and the Feynman plate tricks [12]. As a result, there is only a single label needed to distinguish representations in $\mathbb{R}^4$, and it has only two values ($\mathbb{Z}_2=\{0,1\}$). There are exactly two distinct representations of a genus-one system in $\mathbb{R}^4$.

Finally, in $\mathbb{R}^5$ the single remaining representation label from $\mathbb{R}^4$ is lost and all representations are equivalent. The universal representation lives in this dimension [1]. This progressive diminution of the rich structure of inequivalent representations with increasing dimension is summarized in Table I.

IV. EXTENSION TO HIGHER GENUS

In this section we present the analogous representation theory for three-dimensional dynamical systems with genus greater than one [13]. We justify these results by their similarity to the results in the genus-one case. Proofs of these results are available elsewhere [7].

Genus-g attractors (attractors with g "holes") and the branched manifolds that describe them can be embedded in a three-dimensional manifold by a process of "inflation" [13]. Perform the inflation by surrounding each point in the branched manifold with a small ϵ ball and taking the union of all balls. For sufficiently small ϵ the resulting object is a manifold, specifically a genus-g handlebody. This manifold serves as a proxy for the original unseen phase space. The g holes correspond to the splitting regions of the original flow [3,13], often indicating fixed points [e.g., Lorenz (Fig. 1)].

The boundary of a genus-g handlebody is a genus-g surface called a bounding torus [14]. A genus-one system is bound by *the* torus ($S^1 \times S^1$), and the genus-one handlebody is the solid torus $\mathcal{T}$. We now study how the phase spaces of

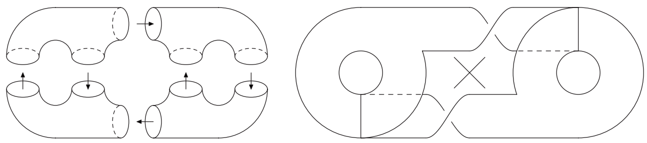

FIG. 1. The Lorenz system: its genus three handlebody phase space (left) and template (right). Arrows indicate the flow directions between trinions and the × denotes the central saddle.

these genus-g attractors can be embedded into $\mathbb{R}^3, \mathbb{R}^4, \ldots$.

As a nontrivial example, the Lorenz system (with typical control parameter values [15]) has a genus three attractor that lives inside a genus three handlebody, which is shown in Fig. 1. The caricature of this flow, its Birman-Williams projection or template [2,3,16], is also shown.

It is possible to construct handlebodies as the union of basic building blocks called *trinions* [17], which are fundamental not only to the topology but also to the dynamics. These trinions are Y junctions, and they come in two types: splitting and joining. These two types correspond to the two fundamental units of templates: splitting and joining charts (see Fig. 2). Trinions are obtained by taking the inflation of either type of chart in the branched manifold. Each trinion has three ports, which are disks D^2 to which the flow is always transverse. Splitting trinions are regions where a flow is split into two separate streams; they have one input port and two output ports. Joining trinions are regions where two separate streams are mixed together; they have two inputs and one output. Just as templates are built "Lego©" style by connecting splitting and joining charts [18], handlebodies are built by connecting trinions. This construction is subject to the following two constraints: (i) output ports of splitting trinions flow to input ports of joining trinions; (ii) output ports of joining trinions flow to input ports of splitting trinions. A genus-g handlebody is created by gluing together $2(g-1)$ trinions: $g-1$ splitting and $g-1$ joining. It is this decomposition of the genus three handlebody into four trinions that is shown in Fig. 1.

We may now enumerate the representations of genus-g systems in $\mathbb{R}^3$. The first representation label is oriented knot type $\mathcal{K}_g$, which is obtained as follows. A genus-g handlebody has a core just as in the genus-one case [7]. To obtain the core, shrink each of the $2(g-1)$ trinions onto a three legged graph or *dreibein* as illustrated in Fig. 3. Each leg carries a flow direction determined by the flow directions through the ports of the trinion. The dreibein for a splitting (joining) trinion has 1 (2) inflowing leg and 2 (1) outflowing legs. The core of the handlebody is the union of the dreibein along common edges. The result is a directed graph of genus g

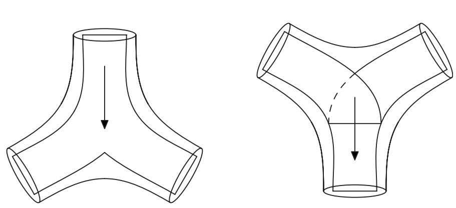

FIG. 2. Trinions with included template charts: splitting (left) and joining (right). The arrow indicates the flow direction.

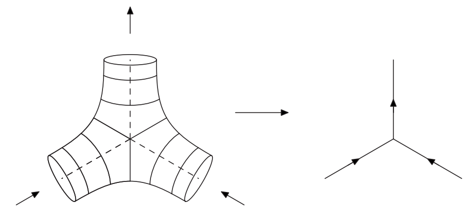

FIG. 3. The core of a trinion (left) is a dreibein (right), obtained by collapsing the indicated disks onto points. Arrows indicate the flow directions. The pictured collapse is for a joining trinion. The collapse for a splitting trinion is obtained by reversing all arrows.

with $2(g-1)$ trivalent vertices and $3(g-1)$ edges. This directed graph represents the fundamental flow directions of the dynamical system, just as the core circle did for genus-one flows. The collection of all knot types $\mathcal{K}_g$ is the set of all embeddings of these genus-g graphs into $\mathbb{R}^3$.

The second label is again parity. A handlebody has an orientation and the mapping $(x,y,z) \mapsto (x,y,-z)$ reverses it. There are exactly two orientations, $\mathbb{Z}_2 = \{\pm 1\}$, just as in the genus-one case [7].

The last representation label is the analog of global torsion. A genus-g handlebody is constructed by gluing $g-1$ splitting and $g-1$ joining trinions together. Altogether, there are $3(g-1)$ such gluings [corresponding to the $3(g-1)$ graph edges above]. The flow is always transverse to the port disks D_i where trinions are glued together. The union of the output disks of the joining trinions may be taken as a global Poincaré section for the flow [13]. The handlebody may be cut along any of these disks and one side rotated $q_i \in \mathbb{Z}$ turns before being reconnected. The result is a spectrum of $3(g-1)$ *local* torsions $(q_1, \ldots, q_{3(g-1)}) \in \mathbb{Z}^{3(g-1)}$.

In three dimensions, in direct analogy with the genus-one case, there is a triple of representation labels $(\mathcal{K}_g, \mathbb{Z}_2, \mathbb{Z}^{3(g-1)})$: the oriented knot type, an orientation, and a spectrum of local torsions [7]. We point out that the problem of distinguishing two knotted circles in $\mathbb{R}^3$ is difficult and still has no general solution. The corresponding problem for higher genus knotted graphs is correspondingly more difficult. Nevertheless, in simple cases it may be reasonable to distinguish embedded graphs "by inspection."

In $\mathbb{R}^4$ many of these distinct representations become equivalent as obstructions to isotopy are lifted [1]. Since graphs are essentially one-dimensional objects, all of their embeddings in $\mathbb{R}^4$ are isotopic, just as all embedded closed curves are isotopic. Knotted graphs become unknotted just as knotted circles do [7]. Oriented knot type is no longer a representation label. Parity also ceases to distinguish embeddings.

Once again, local torsion is more subtle [7]. We anticipate that, in analogy with the genus-one case, the local torsions at each of the $3(g-1)$ ports fall into two classes: q_i even and q_i odd. Thus, the integer $\mathbb{Z}$ that characterized the torsion at each port is reduced to $\mathbb{Z}_2 = \{0,1\}$. But this is not all. On any trinion, a single twist on any port can be translated into a pair of twists—one in each of the other two ports—at the expense of introducing a writhe or twisting of the legs near those ports. However, this writhing is easily pulled apart in $\mathbb{R}^4$ (see Fig. 4). This means that a single twist on any one port is fully

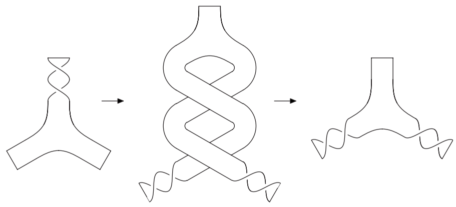

FIG. 4. Conversion of a twist into two twists plus a writhe. The writhe is removed in $\mathbb{R}^4$ by an isotopy.

interchangeable with a pair of twists, one in each of the other two ports. In other words, out of the three twists on each of the three ports of the trinion, only two are now independent. Instead of there being $3(g-1)$ $\mathbb{Z}_2$ local torsions, there are only $2(g-1)$. We conclude that in $\mathbb{R}^4$ there is only one representation label, a spectrum of $2(g-1)$ local torsions, $(q_1,\ldots,q_{2(g-1)}) \in \mathbb{Z}_2^{2(g-1)}$.

Finally, in $\mathbb{R}^5$ all representations become equivalent [7]. The remaining local torsions become isotopic to the identity exactly as in the genus-one case. We arrive at a universal representation for genus-g dynamical systems in five dimensions, which is two dimensions lower than that guaranteed by Wu's theorem $(2\times 3+1=7)$. The complete representation theory for genus-g systems is summarized in Table II.

V. SUMMARY

We have taken the first steps in creating a representation theory for dissipative dynamical systems. These steps have been motivated by and guided to some extent by an analogy with the representation theory of groups. Representations are embeddings of an n-dimensional dynamical system into $\mathbb{R}^k$, $k\geq n$. Equivalence of representations is by isotopy. This theory has been carried out only for three-dimensional dissipative dynamical systems. The program has yet to be implemented for higher-dimensional dynamical systems, $n>3$, since the topological nature of suitable invariants in higher dimensions is not yet known.

The general program is as follows. Identify all the labels necessary to distinguish among inequivalent representations of the n-dimensional dynamical system when mapped into $\mathbb{R}^k$ for $k=n$. These labels are obstructions to isotopy. When inequivalent representations are mapped into a Euclidean space of one higher dimension, some may become equivalent because there is more room to "move around" and so avoid certain obstructions. As a result, some representation labels are no longer necessary. This process continues until all representations are isotopic and all obstructions have vanished. Then there is only one universal representation. According to a result of Wu, this universal representation definitely exists in $\mathbb{R}^{2n+1}$. However, it may also exist in lower dimensions: five for three-dimensional systems.

TABLE II. Representation labels for arbitrary genus systems. For parity $\mathbb{Z}_2=\{\pm 1\}$, while for local torsion $\mathbb{Z}_2=\{0,1\}$. $\mathcal{K}_g$ denotes the set of oriented knotted graphs of genus g.

	Obstructions to isotopy		
Representation labels	$\mathbb{R}^3$	$\mathbb{R}^4$	$\mathbb{R}^5$
Local torsion	$\mathbb{Z}^{3(g-1)}$	$\mathbb{Z}_2^{2(g-1)}$	
Parity	$\mathbb{Z}_2$		
Knot type	$\mathcal{K}_g$		

The representation theory for three-dimensional dynamical systems has been developed in two steps. In an earlier contribution we studied the representation theory of dynamical systems of "genus-one" type. This class includes nonautonomous dynamical systems such as periodically driven two-dimensional nonlinear oscillators and autonomous three-dimensional dynamical systems such as the Rössler attractor. All these systems have the solid torus $\mathcal{T}$ as their natural phase space, and their chaotic dynamics is generated by "stretching and folding." For these systems there are three representation labels in $\mathbb{R}^3$: knot type, parity, and global torsion; one in $\mathbb{R}^4$: global torsion (mod 2) and none in $\mathbb{R}^5$. The universal representation exists in $\mathbb{R}^k$, $k\geq 5$. This information is summarized in Table I.

The present work completes this representation theory for all three-dimensional "genus-g dynamical systems" and their attractors. This class includes the Lorenz attractor and other systems whose chaotic dynamics are generated by "tearing and squeezing." These systems have genus-g handlebodies as phase space. For these systems there are three representation labels in $\mathbb{R}^3$: knot type, parity, and a spectrum of local torsions; one in $\mathbb{R}^4$: a reduced spectrum of local torsions (mod 2) and none in $\mathbb{R}^5$. The universal representation exists in $\mathbb{R}^k$, $k\geq 5$. This information is summarized in Table II.

These results allow us to conclude that some of the information obtained by analyzing a three-dimensional embedding of a three-dimensional dynamical system depends on the embedding (knot type, parity, and global or local torsion) and some are embedding independent. However, since the universal embedding exists in $\mathbb{R}^5$ (or higher), any information extracted from such an embedding must be intrinsic—it must depend on the dynamics alone and not at all on the embedding.

ACKNOWLEDGMENT

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