2014

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JOURNAL OF MATHEMATICAL PHYSICS **55**, 123504 (2014)

When the charge on a planar conductor is a function of its curvature

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(Received 21 July 2014; accepted 13 November 2014; published online 10 December 2014)

While there is no general relationship between the electric charge density on a conducting surface and its curvature, the two quantities can be functionally related in special circumstances. This paper presents a complete classification of two-dimensional conductors for which charge density is a function of boundary curvature. Whenever the curvature function is non-injective, the conductor must transform under one of the planar symmetry groups. In particular, for the charge density on a closed conductor with smooth boundary to be a function of its curvature, the conductor must possess dihedral symmetry with a mirror line running through each curvature extremum. Several examples are presented along with explicit charge-curvature functions. Both increasing and decreasing functions were found. © *2014 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4903168]

I. INTRODUCTION

It is known that there is no general mathematical relationship between the surface charge density σ and curvature κ on a conducting surface. In particular, it is not generally possible to write $\sigma(\kappa)$ or $\kappa(\sigma)$ as single valued functions over the entire surface. Indeed, explicit examples are known in which σ and κ have misaligned extrema, forcing any mathematical function between them to be multiple-valued.[1] On the other hand, it is certainly possible for the two quantities to be related one-to-one in certain cases. For example, it is known that $\kappa \sim \sigma^4$ for all of the quadratic surfaces (ellipsoids, paraboloids, hyperboloids, etc.).[2,3] The quadratic surfaces are geometrically very nice (in particular, they are highly symmetric), so it is natural to conjecture that some geometrical niceness is necessary for the existence of a functional relationship between charge and curvature. In this paper, we determine all two-dimensional conductor geometries for which charge density is a function of boundary curvature. The geometrical niceness condition is symmetry: the conductor must be invariant under one of the planar symmetry groups. This paper does not investigate two-dimensional surfaces in three dimensions, but we conjecture that symmetry plays an essential role there as well.

The essential problem is that while curvature is a local phenomenon, the charge distribution is global, determined by solving the Laplace equation. To see more clearly how this is an issue, consider two spheres of different radii held at a common potential. Ignoring their mutual interaction, the problem is trivial: the charge density is constant on each surface, and we can conclude that $\kappa \sim \sigma$ (and are locally constant). However, a complete analysis that includes their mutual interaction shows that the spheres polarize each other at any finite separation, leading to varying σ at constant κ, precluding a functional relationship of the form $\sigma(\kappa)$.[4,5] The existence of a functional relationship is not stable under perturbation.

It is difficult to analyze the general situation with the curvature and charge density depending so differently on the geometry. Section II examines the electrostatic problem from a geometrical perspective and presents expressions for σ and κ in terms of the conformal factor of the geometry. These expressions are used in the following sections to constrain the possible geometries possessing functional relationships: Section III shows that if the conductor has constant curvature anywhere, it has constant curvature everywhere; Section IV considers curves of monotonic curvature; and Sec. V shows that in the general case of non-injective curvature, the conductor must be invariant under a planar symmetry group. In Sec. VI, the classification of planar symmetry groups is used to enumerate all possible conductor geometries possessing functional relationships. The main result is that a closed conductor with smooth boundary must possess dihedral symmetry with a mirror line through every curvature extremum. Section VII exhibits explicit conductor geometries of various symmetry types along with their charge-curvature functions. Conclusions are stated in Sec. VIII.

II. ELECTROSTATICS AND GEOMETRY

In this section, we frame the electrostatic problem in a geometrical context, deriving expressions for the curvature κ and charge density σ in terms of the conformal factor Ω. Consider a conductor surface S with translational invariance in the z-direction, so that we can write $S = \gamma \times \mathbb{R}$, where γ is a collection of disjoint simple curves in the xy-plane. These curves need not be smooth: corners are allowed. We are mostly interested in the case where the curves are closed and bound a finite region of the plane, though we will also consider unclosed curves that either bound infinite regions or no region at all. The curvature κ of the surface is the curvature of γ as a plane curve (surface mean curvature).

On γ we have $\sigma = \epsilon E$, where ϵ is the permittivity (henceforth set to 1) and E is the electric field strength, which is obtained from the potential ϕ by $E = \|\nabla\phi\|$. In the plane outside γ, ϕ satisfies the Laplace equation $\nabla^2\phi = 0$, and its level surfaces are the equipotentials. Define coordinates (u, v) adapted to this geometry, where u parametrizes the equipotentials, and v is a coordinate along each equipotential. These coordinates need not be defined inside the conductors. It is a result of differential geometry[6] or of complex analysis (Riemann Mapping Theorem) that the mapping $(x, y) \mapsto (u, v)$ can always be chosen to be conformal (though this is only guaranteed away from the boundary, γ). The metric thus has the form

$$ds^2 = \Omega(du^2 + dv^2),$$

where $\Omega = (\partial_u x)^2 + (\partial_v x)^2 = (\partial_u y)^2 + (\partial_v y)^2$ is the conformal scale factor. The non-zero components of the metric tensor are $g_{uu} = g_{vv} = \Omega$, $(g^{uu} = g^{vv} = \Omega^{-1})$, and its determinant is $g = \Omega^2$. Note that x and y are subject to the Cauchy-Riemann conditions ($\partial_u x = \partial_v y$ and $\partial_v x = -\partial_u y$) from which it follows that x and y are harmonic in u and v.

In this metric, the Laplacian is

$$\nabla^2 = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right) = \Omega^{-1}\left(\partial_u^2 + \partial_v^2\right),$$

so that the Laplace equation for ϕ is just $\partial_u^2\phi + \partial_v^2\phi = 0$. By construction of the adapted coordinates $\partial_v\phi = 0$, so that $\phi = c_1 u + c_2$, which was to be expected since u and ϕ both parametrize the surfaces of constant ϕ. Without loss of generality, we may identify $\phi \equiv u$.

The covariant components of the electric field vector are $E_i = -\partial_i\phi = (-1, 0)$, hence its length is $E = \|E_i\| = \sqrt{g^{ij}E_iE_j} = \sqrt{g^{uu}} = \Omega^{-1/2}$, and

$$\sigma = E = \Omega^{-1/2}. \tag{1}$$

The charge density depends only on the conformal factor.

The curvature is the tangential component of the change in the unit normal along the curve, $\kappa = t_i t^j \nabla_j n^i$, with ∇ the covariant derivative. In the adapted coordinate system, the covariant components of these unit vectors are $t_i = (0, \Omega^{1/2})$ and $n_i = (\Omega^{1/2}, 0)$. Written out

$$\kappa = t_i t^j\left(\partial_j n^i + \Gamma^i_{jk}n^k\right) = \Gamma^v_{vu}n^u. \tag{2}$$

The sole non-zero Christoffel symbol is

$$\Gamma^v_{uv} = \frac{1}{2}g^{vv}\partial_u g_{uu} = \frac{1}{2}\partial_u \ln\Omega,$$

and substituting this into Eq. (2) yields

$$\kappa = \frac{1}{2}\Omega^{-3/2}\partial_u\Omega. \tag{3}$$

The curvature depends on the conformal factor and its normal derivative.

Equations (1) and (3) formalize the basic problem: σ and κ depend differently on the geometry. However, having both quantities expressed in terms of Ω allows a general solution to the problem and a means to explicitly construct $\sigma(\kappa)$ whenever Ω is known (see Sec. VII).

Finally, note that we can rewrite Eq. (3) using Eq. (1) as

$$\kappa = \frac{1}{2}\Omega^{-3/2}\partial_u\Omega = -\partial_u\Omega^{-1/2} = -\partial_u E,$$

obtaining the two-dimensional analog of the well known result in three dimensions that the field *gradient* always tracks the local curvature.[7–9]

III. SEGMENTS OF CONSTANT CURVATURE

In this section, we show that if κ is constant on some segment $\gamma_0 \subset \gamma$, then it is constant everywhere, and γ is a geometrical circle (or line). Suppose that κ is constant on γ_0. If a relation $\sigma(\kappa)$ is to exist, then σ must also be constant on that segment. But then Ω is constant by Eq. (1), that is, $\partial_v\Omega = 0$. Using complex coordinates $z = x + iy$ and $w = u + iv$, the conformal factor is $\Omega = z'\bar{z}'$. Writing $\partial_v = i(\partial_w - \partial_{\bar{w}})$, we find that $\partial_v\Omega = 0$ becomes $z''\bar{z}' = z'\bar{z}''$, which separates to $z''/z' = \bar{z}''/\bar{z}' = \lambda$, with λ real. If $\lambda = 0$, then $z'' = 0$ and $z(w)$ is a linear transformation, whereas for $\lambda \neq 0$, this yields the exponential map $z = a + b\exp(\lambda w)$. Either way $z(w)$ is entire and extends to all w. The first case yields straight line conductors and the second geometric circles (with the trivial relationship $\sigma \sim \kappa = \text{const.}$).

IV. MONOTONIC CURVATURE

If the curvature κ is monotonic along the curve γ, then by Sec. III, we may assume that it is strictly monotonic, and a function expressing $\sigma(\kappa)$ necessarily exists. However, having monotonic curvature is an extremely restricting condition. A theorem of Tait[10] and Kneser[11] states that if curvature is non-zero and monotonic along a curve, then its osculating circles are nested.[12] It immediately follows that the curve itself is bounded in the direction of increasing (absolute value of) curvature, trapped within each consecutive osculating circle. A complete characterization of such curves is thus (cf. Poincaré-Bendixon): in the direction of increasing curvature, either (i) the curve is finite and ends at some point or (ii) the curve is an infinite spiral that either (iia) limits to a fixed point or (iib) limits to a circle (limit cycle). In the direction of decreasing curvature, either (i) the curve is finite and ends at a point or (ii) the curve is infinite and (iia) continues spiraling out to infinity or (iib) is asymptotic to a limit cycle. We note that the curve could be asymptotic to a straight line, which is a special case of (iia). All possible geometries are sketched in Fig. 1. Note that none of these can bound any region in the plane, finite or infinite.

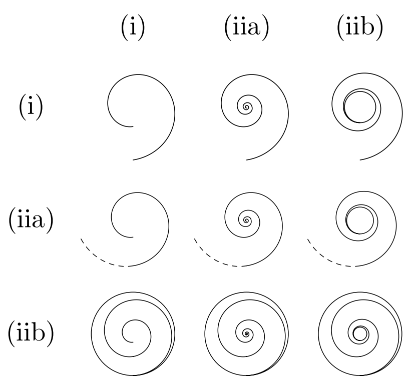


FIG. 1. The nine basic one dimensional conductor types. Dashed lines indicate continued spiraling out to infinity. The columns (rows) correspond to the forward (backward) behavior of the curve.

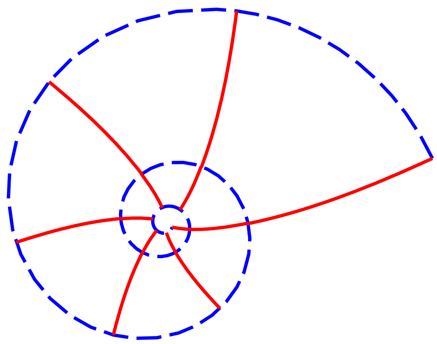


FIG. 2. Equipotentials (dashed blue) and field lines (solid red) for the logarithmic spiral geometry defined by Eq. (4) with $\alpha = \exp(i\pi/16)$.

An explicit conductor geometry representing type (iia)-(iia) is a logarithmic spiral (see Fig. 2). Writing $z = x + iy$ and $w = u + iv$, this solution is generated by the conformal map

$$z = \exp(\alpha w), \tag{4}$$

where α is a complex constant. The equipotentials—(constant u) are field lines, (constant v) are counter-rotating spirals. A quick calculation shows $\Omega \sim |z|^2$, $\partial_u \Omega = \Omega$, and that $\sigma \sim \kappa \sim 1/|z|$.

If the curvature is monotonic and attains zero, then it necessarily attains negative values as well. Applying Tait-Kneser to the two halves of the curve separated by the flat point (zero curvature) shows that each side is either (i) finite or (ii) limits to (iia) a fixed point or (iib) a limit cycle. All six possible geometry combinations can be constructed by taking any two examples from the first row in Fig. 1, rotating one by π, and gluing them at their flat points. An example of type (i-iia)-(i-iia) is the Cornu spiral (for which curvature is proportional to arc length, $\kappa \sim s$), and is shown in Fig. 3.

If curvature discontinuities (corners) are allowed, then it is possible to form conductors with monotonic curvature that bound either infinite (scimitar shape) or finite (crescent shape) regions, since the discontinuities allow positive and negative curvature to coexist without flat points or local extrema. In the case of a scimitar, curvature is largest in absolute value near the discontinuity, the curvature approaches zero toward infinity, and the curve is asymptotically flat. Figure 4 exhibits examples of each type.

V. NON-MONOTONIC CURVATURE

In this section, the curvature κ will be non-monotonic along γ. In particular, κ can have local extrema, allowing for closed conductors without corners. The existence of functional relationships is no longer automatic. In fact, we show that only symmetrical conductor geometries possess these relationships. As per the results of Sec. III, we assume that the curvature is never constant on any finite segment $\gamma_0 \subset \gamma$.

Suppose for $x_0 \neq y_0 \in \gamma$ that $\kappa(x_0) = \kappa(y_0)$ and that neither point is an extremum of curvature so that there are neighborhoods γ_x of x_0 and γ_y of y_0 on which the curvature is monotonic and takes the same range of values: for each $x \in \gamma_x$ there is a unique $y \in \gamma_y$ such that $\kappa(x) = \kappa(y)$. The existence of a function $\sigma(\kappa)$ then requires that $\sigma(x) = \sigma(y)$. But then from Eq. (1), it follows that $\Omega(x) = \Omega(y)$. Since arc length along γ is given by $\int ds = \int \Omega(x)dx$, the arc length from x_0 up to x

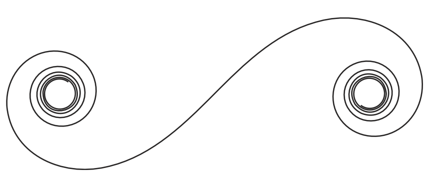


FIG. 3. Cornu spiral as an example of a type (i-iia)-(i-iia) conducting filament.

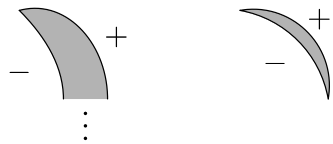


FIG. 4. Examples of curves with monotonic, but not everywhere continuous, curvature bounding infinite (left) and finite (right) regions of the plane.

in γ_x and from y_0 up to y in γ_y are equal. Thus, the curves have equivalent curvature as a function of arc length, and by Frenet-Serret the curves are isometric: there is some symmetry transformation T taking γ_x to γ_y.

Let f be the conformal map defining the conductor geometry, and let Γ_x and Γ_y be the preimages of γ_x and γ_y, respectively. The map $S = f^{-1}Tf$ is isogonal and maps $\Gamma_x \to \Gamma_y$. The two maps f and TfS are analytic and agree on Γ_x, so they are the same map. It follows that T is a global symmetry of f and of γ.

Now suppose that $p \in \gamma$ is a local extremum of the curvature. In a neighborhood of p, write $\gamma = \gamma_1 \cup \gamma_2$ ($\gamma_1 \cap \gamma_2 = p$), where κ is monotonic and has the same range of values on each γ_i. Applying the previous argument, we find an isometry T that takes $\gamma_1 \to \gamma_2$, but that also fixes p. Thus, T must be either a rotation or a reflection. Since p is a point of extreme curvature, γ must lie to one side of the tangent line t through p (in a sufficiently small neighborhood). In particular, T must preserve the half-planes on either side of t, and so must be a reflection. The fixed point set l of T is a line perpendicular to t, and each point in $l \cap \gamma$ is necessarily a point of extreme curvature (when κ is continuous there).

Thus, any two points of equal curvature are connected by a planar isometry T fixing γ setwise. T is an element of the plane Euclidean group $E(3)$, so is either a translation, rotation, reflection, or glide. The set of all symmetries of γ will form one of the planar symmetry groups. Note that orbits under translations and glides are unbounded, so that in these cases, γ necessarily extends to infinity, but can be composed of either a finite or an infinite number of connected components.

VI. CONDUCTORS AND PLANAR SYMMETRIES

It has been established that any curve γ with non-monotonic curvature must be invariant under one of the planar symmetry groups. All possible symmetry types are trivially realizable by appropriately tiling conductors with monotonic curvature (either filaments or crescents, see Sec. IV), but smooth closed curves are more restrictive. A fundamental restriction is given by the four-vertex theorem: any smooth closed curve has at least four points of maximal curvature (two maxima, two minima). In the present context, this means that a smooth closed curve will have at least two reflection symmetries (one through the curvature maxima, the other through the minima), so these shapes require symmetry groups with at least two intersecting mirror lines. This rules out monotonic eggs: reflection symmetric closed curves with monotonic curvature on the fundamental domain. One can, however, construct a monotonic egg with corners: take any arc with monotonic curvature that lies to one side of the line connecting its end points and generate a closed curve by reflecting the arc across this line. In the sequel, egg will mean a monotonic egg with corners.

A. Point (Rosette) groups

The point groups have no translation subgroups, consisting entirely of rotations and reflections. The simplest symmetry is a single reflection, type $*1\bullet \equiv *$ (orbifold notation is used throughout) and can be realized in several different ways: (1) hyperbola-like conductors that are asymptotically flat with a single curvature minimum on the symmetry axis (curvature profile $(0, \kappa^*]$); (2) cusp-like conductors with a corner on the symmetry axis (profile $(\kappa_*, 0)$); and (3) eggs (profile (κ_*, κ^*)). The first two bound infinite regions, the third finite. Moreover, any number of these elements can be aligned on their common symmetry axis so long as the range of curvature values obtained along

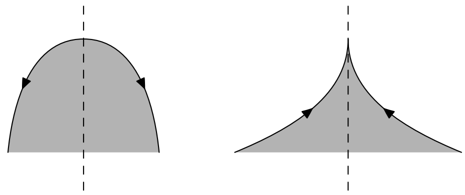


FIG. 5. Examples of symmetry type $*$ with curvature profiles $(0, \kappa^*]$ (left), $(\kappa_*, 0)$ (center), and (κ_*, κ^*) (right). Arrows point in the direction of decreasing curvature. The first two bound infinite regions of the plane.

each are disjoint, assuming one wants a single global functional relationship. If it is only desired to have a separate functional relationship on each component of γ, then there is no restriction on the curvature ranges of each component, though they must still be aligned on a common symmetry axis. Examples of each type are shown in Fig. 5.

Next are the higher order dihedral groups $*N\bullet \equiv *N$, $N \geq 2$. These are the only point groups with at least two intersecting mirror lines (N of them). Since isolated smooth closed conductors must have at least two mirror lines, *charge is a function of curvature for an isolated smooth closed conductor if and only if it has dihedral symmetry with a mirror line through every curvature extremum*. The simplest possible shape is the ellipse, which belongs to $N = 2$. Multiple component conductors with dihedral symmetry can be constructed by tiling conductors with $*$ symmetry, e.g., multipole lenses. Explicit charge-curvature relationships are given in Sec. VII for both possibilities.

The last of the point groups are the cyclic groups $N\bullet$, $N \geq 2$, consisting only of rotations. Since these groups have no reflections, all bounding conductors realizing them must have corners. A basic representative in this class can be constructed by modifying the egg construction: instead of flipping the monotonic arc about the line segment joining its endpoints, rotate it by π about the midpoint of that segment. Two examples are shown in Fig. 6 for $N = 2$.

B. Line (Frieze) groups

The line groups have one translation subgroup, so describe patterns that repeat periodically in one dimension. The groups $\infty\infty$ and $\infty\times$ have no mirror lines, so their representatives must contain corners. The groups $\infty*$, $*\infty\infty$, and $*22\infty$ contains mirrors, but none intersect. These can be realized with smoother shapes than the first collection, but closed curves must still contain at least one corner. Finally, there is $2*\infty$, which has a pair of intersecting mirrors and can be realized by periodically tiling identical smooth conductors with $*2$ symmetry along a common mirror axis (e.g., a periodic sequence of ellipses). Figure 7 depicts examples of type $\infty\times$, which is generated by a glide. Representatives of other frieze groups can be generated quite similarly.

C. Space (Wallpaper) groups

These include the mirrorless groups $\circ$, 632, 442, 333, 222, $\times\times$, and $22\times$; the non-interesting mirror groups $22*$, $**$ and $*\times$; and the intersecting mirror groups $4*2$, $3*3$, $2*22$, $*632$, $*442$, $*333$, and $*2222$. All space groups have two independent translation subgroups, so special considerations

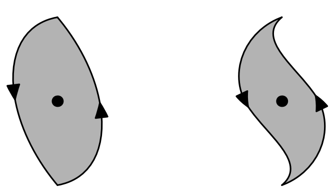


FIG. 6. Examples of symmetry type $2\bullet$ with curvature profiles (κ^*, κ_*). The left curve has everywhere positive curvature, while the right has positive and negative curvatures separated by a flat point.

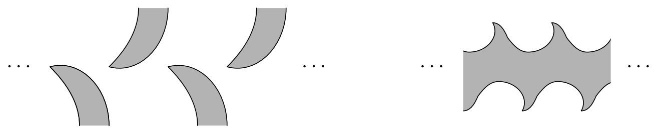


FIG. 7. Conductors transforming under ∞×, which is generated by a glide reflection. Left: an infinite number of components. Right: a single component extended to infinity.

apply. Modding out both translations takes the plane onto a torus, a closed manifold, and the total electric charge necessarily vanishes

$$q_{\mathrm{tot}} = \int_M \rho \, dA = \int_M \nabla \cdot \vec{E} \, dA = \int_{\partial M} \vec{E} \cdot d\vec{n} = 0,$$

as the boundary is empty. In order to avoid the trivial solution (no charge) and ghost charges,[6] it must be possible to find a fundamental translational domain with a residual two-fold symmetry according to which half of the conductors can be given positive charge and the symmetry related half negative charge. That is, the tessellation must be two colorable: the symmetry group G must have a subgroup (color kernel) K fixing the colors. Most wallpaper groups are two-colorable in more than one way, and only 333 is not two-colorable at all (see Table I).[13] Every possibility can be realized by tiling asymmetric filaments or crescents. Note, however, that in this case, σ will be two-valued, but $|\sigma|$ will be single-valued (and invariant under K) with operations in $G - K$ taking $\sigma \to -\sigma$.

Tiling symmetric conductors (eggs, ellipses, etc.) introduces a further restriction: not only must K fix the color of the conductors, it must fix the conductors themselves according to their self-symmetries. This *tile group* T is a point subgroup of K that is either $*$ or a maximal point subgroup of G (there is always a unique maximal point subgroup T_G of G containing T). Group operations of T_G not in T must swap colors and thus conductors. If T is not $*$, then it fixes a unique point p, so there would be (at least) two distinct conductors containing p, which is a contradiction. However, if T is a mirror then there is no unique fixed point, so separate (symmetry related) fixed

TABLE I. Wallpaper groups, two-fold color kernels, and tile groups.

Group	Color kernel	Tile group
*632	*333	*3
	3*3	*
	632	•
4*2	442	4•
	2*22	*2
	22×	•
2*22	*2222	*2
	22*	2•, *
	2222	2•
	*×	*
	22×	•
*442	*442	*4
	4*2	*
	*2222	*2
	2*22	*
	442	•
*2222	*2222	*2
	2222	•
	22*	*
	**	*
	2*22	*2

Group	Color kernel	Tile group
*333	333	•
3*3	333	3•
333	∅	∅
632	333	3•
442	442	4•
	2222	2•
2222	∘	•
	2222	2•
22×	2222	2•
	××	•
*×	**	*
	∘	•
	××	•
**	**	*
	*×	*
	∘	•
	××	•
22*	××	•
	**	*
	22×	2•
	2222	2•
	22*	2•, *

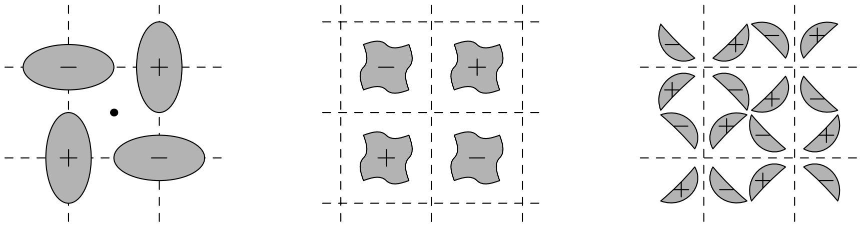


FIG. 8. Conductors having symmetry type 4∗2 and color types 2∗22 (left), 442 (center), and 22× (right). Individual conductors have symmetry types (tile groups) ∗2 (left), 4• (center), and • (right). Positively and negatively charged conductors are indicated by + and −, respectively.

points can be chosen for each conductor. To determine the tile groups, for each maximal point subgroup T_G of G, check if $T_G \subset K$. If so, $T = T_G$ is a tile group. If $T_G \not\subset K$, then if T_G contains a mirror subgroup contained in K, then $T = *$. Otherwise, T is just the identity, and we write $T = \bullet$. For example, consider $G = *632$, which has three maximal point subgroups (up to conjugation): $*6$, $*3$, and $*2$. For $K = *333$, only $*3 \subset G$ is a subgroup of K, and this is the tile group. For $K = 3*3$, none of the subgroups are in K ($*3 \subset K$ is a subgroup of $*6 \subset G$), but $* \subset *3 \subset K$, so the tile group is $T = *$. Finally, for $K = 632$ no subgroup of G is in K, and no point subgroup of K contains a mirror, so $T = \bullet$. A trivial tile group of $\bullet$ can only be produced with (asymmetric) filaments and crescents. Notably, there are only three distinct tile groups allowed by seven color types: $*4$ ($*442/*442$), $*3$ ($*632/*333$), and $*2$ ($4*2/2*22$, $2*22/*2222$, $*442/*2222$, $*2222/*2222$, and $*2222/2*22$). Examples of all three color types of $4*2$ are shown in Fig. 8.

We end this section with a numerical example illustrating the effect of symmetry (and its lack) on the charge distribution. Figure 9 shows two square conductors with opposite total charge in the torus. Each square conductor has dihedral $*4$ symmetry. The conductors are equally spaced horizontally so that the vertical mirror symmetries of the squares always extend to the whole pattern (so the left and right halves of each square always have the same charge density), but they have varying vertical separation, so that the remaining symmetries do not (typically) extend. The parameter ξ measures this vertical offset as a fraction of the height of the torus (ξ has period 1). For almost all values of ξ, the pattern has symmetry $22*$. However, when $\xi = 0$, the horizontal mirror lines of each square extend to the pattern (type $*2222$), and for $\xi = 0.5$, all mirror lines extend (type $*442$). The dihedral mirror lines cut the square in eight places, cutting the boundary into eight segments. Since the charge is always left-right symmetric, we only track the fraction of the total charge on the four left segments as a function of ξ as shown in the bottom of Fig. 9 (charge distributions were computed via relaxation). Typically, all four segments have different charge, but for $\xi = 0$ the charge becomes equal on pairs of symmetry related segments, and for $\xi = 0.5$, all four segments have equal charge, and all four are symmetry related. While the plot indicates an equality of charge at $\xi \approx 0.2$, this is accidental: while the total charge on a pair of segments happens to be equal, the charge is not equal pointwise.

VII. SOME EXPLICIT CHARGE-CURVATURE FUNCTIONS

Section II established how charge density and curvature depend on the conformal factor and its normal derivative. These relationships can be used to generate an analytic criterion for the existence of a charge-curvature relationship: *the charge density σ and curvature κ are functionally related precisely when $\Omega^{-3/2}\partial_u\Omega$ and $\Omega^{-1/2}$ are functionally related at constant potential $\phi = u$.*

The necessary and sufficient condition for the functional dependence of two functions f and g of variables u and v is the vanishing of the determinant of the partial derivative matrix $\partial(f,g)/\partial(u,v)$. However, this is too strong a condition here since we only require a functional relationship to exist for constant u, not variable u, and this would eliminate many interesting solutions. To circumvent this restriction, we first make a specific *Ansatz* for the form of $\partial_u\Omega$, then enforce functional dependence and solve the resulting partial differential equation (PDE). The two general

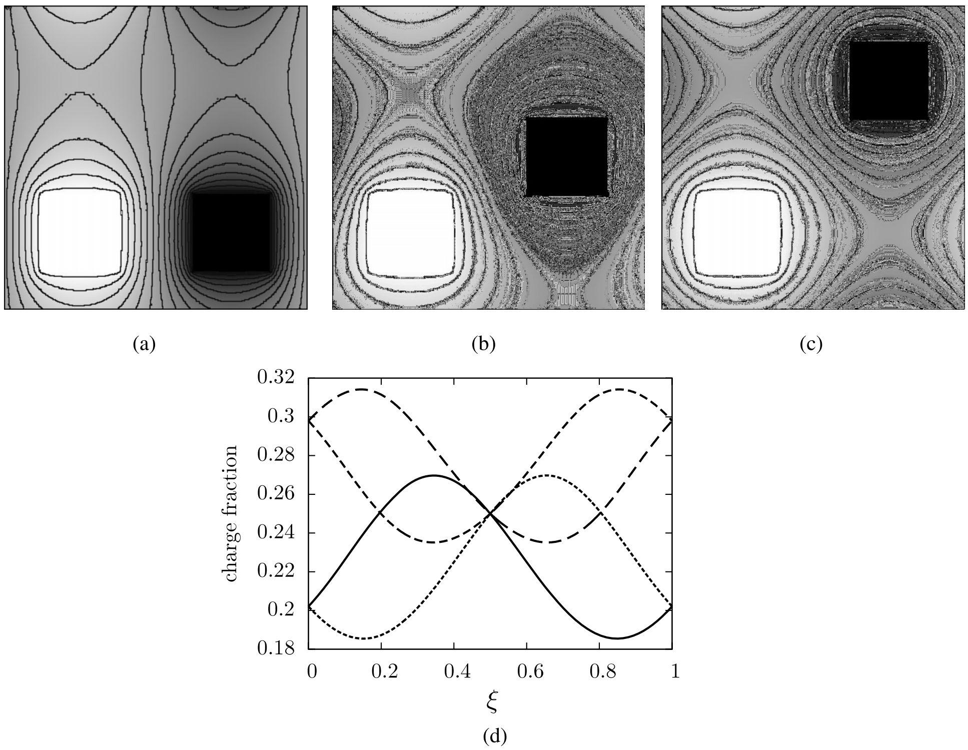


FIG. 9. (a), (b), and (c) Equipotentials for two square conductors held at opposite potentials with periodic boundary conditions and varying vertical offsets ξ. (a) ($\xi = 0$) has symmetry type $*2222/*2222$, (b) ($\xi = 0.25$) has type $22*/**$, and (c) ($\xi = 0.5$) has type $*442/*442$, consistent with the dihedral $*4$ symmetry of each square. (d) The charge on each segment of the square between mirror lines defined by the dihedral symmetry. Since charge distributions are always left/right symmetric, only the charge on the four left segments are plotted and as fractions of half the total charge. The closer the symmetry is to supporting the full dihedral symmetry, the more the charge distributions agree. The charge difference between the two squares is of order 10^{-7} for all ξ, consistent with zero.

Ansätze we consider are (1) $\partial_u\Omega = f(u)$ and (2) $\partial_u\Omega = f(\Omega)g(u)$, where f and g are to be determined. Though (1) is a special case of (2), we treat it explicitly since its solutions are of independent interest.[14]

Ansatze 1: $\partial_u\Omega = f(u)$.

If $\partial_u\Omega = f(u)$, then $\kappa = \frac{1}{2}\Omega^{-3/2}f(u)$, and it follows from Eq. (1) that at constant u

$$\kappa \sim \Omega^{-3/2} \sim \sigma^3.$$

As we will show, it is no accident that this resembles the relationship $\kappa \sim \sigma^4$ that holds for the quadratic surfaces in three dimensions.

To determine the geometry, first recall that $\Omega = (\partial_u x)^2 + (\partial_v x)^2$, where x is harmonic. If $\partial_u\Omega = f(u)$, then $\partial_u\Omega$ must be independent of v, or $\partial_v\partial_u\Omega = 0$. Hence, we seek functions $x(u,v)$ that satisfy Laplace's equation, subject to the condition that $\partial_v\partial_u\Omega = 0$. While the linearity of Laplace's equation enables a direct approach, this problem has already been solved. Specifically, having $\partial_v\partial_u\Omega = 0$ for a coordinate system (u,v) is precisely the condition under which the Helmholtz equation for an arbitrary function $F(u,v)$ separates.[15] The only coordinate systems with this property are the Cartesian and the conic based—parabolic, polar, and elliptic coordinates. Thus, all conic conductors (quadratic curves) have $\kappa \sim \sigma^3$. Examples are shown in Fig. 10. The hyperbolic/elliptic case has symmetry type $*2$, while the parabolic case has symmetry type $*$. In the case of hyperbolic equipotentials, a symmetry related pair of equipotentials must be chosen as the conducting surfaces to ensure both symmetry and a functional relationship.

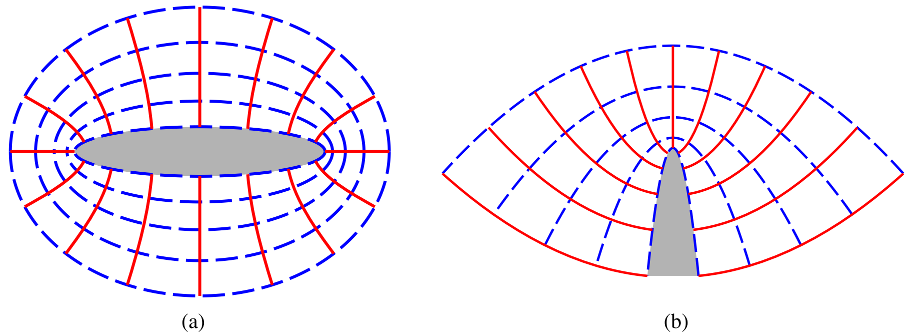


FIG. 10. Equipotentials (dashed blue) and field lines (solid red) for elliptical ((a), type $*2$) and parabolic ((b), type $*$) conductors. The equipotentials and field lines can be swapped for the elliptical conductors, giving rise to pairs of hyperbolic conductors.

Ansatze 2: $\partial_u \Omega = f(\Omega) g(u)$.

Writing this as $\partial_u \Omega / g(u) = f(\Omega)$, we require the functional dependence of $\partial_u \Omega / g(u)$ and Ω. The functional dependence condition yields the equation

$$\frac{\partial_u g}{g} = \frac{\partial_v \Omega \partial_{uu} \Omega - \partial_u \Omega \partial_{uv} \Omega}{\partial_u \Omega \partial_v \Omega}.$$

The L.H.S. is independent of v, so taking the partial derivative with respect to v eliminates g, and after simplification yields the following PDE for Ω:

$$\partial_u \Omega \partial_v \Omega \left(\partial_v \Omega \partial_{uuv} \Omega - \partial_u \Omega \partial_{uvv} \Omega \right) = \partial_{uv} \Omega \left(\partial_v \Omega^2 \partial_{uu} \Omega - \partial_u \Omega^2 \partial_{vv} \Omega \right). \tag{5}$$

This equation is solved in the Appendix. Here, we describe in detail three particular solutions obtained there.

The first solution is (with $z = x + iy$ and $w = u + iv$)

$$z(w) = w^n, \tag{6}$$

with $0 < n < 2$. Solutions with $n > 1$ describe outer corners of various angles, while solutions with $n < 1$ describe inner corners ($n = 1$ is an infinite straight line). An example of each type is shown in Fig. 11. Both of these forms have symmetry type $*$, however, the specific form $n = 2/p$, with p an integer greater than 2, can describe a p-pole lens of symmetry type $*p$ via tiling. The choice $p = 4$ ($z = \sqrt{w}$) yields the well-known quadrupole lens.[16]

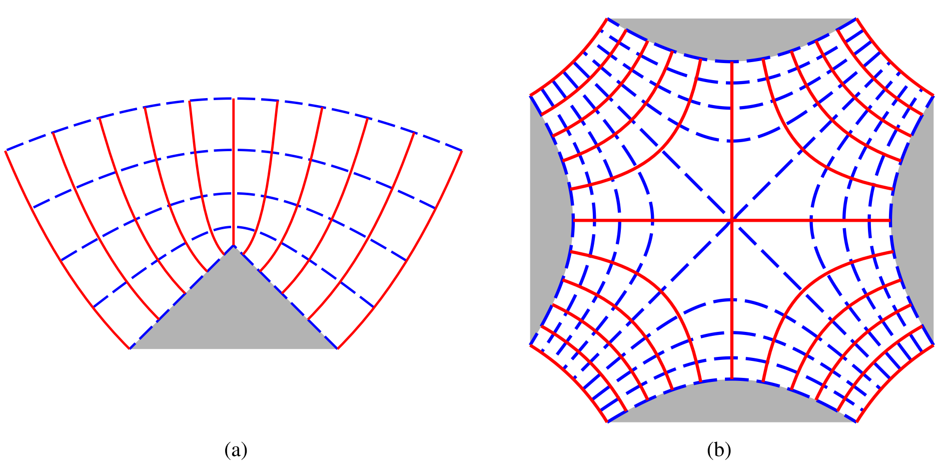
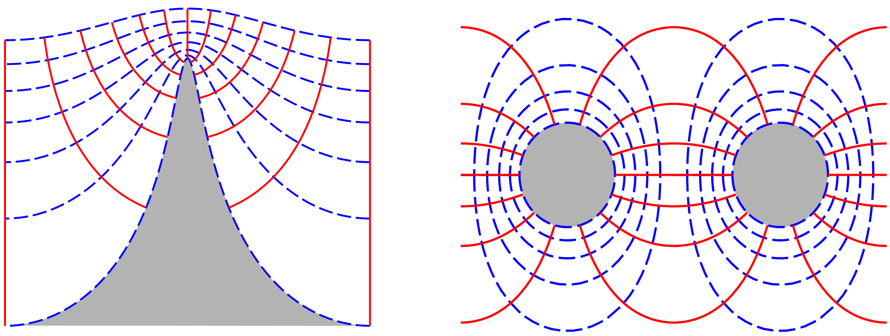


FIG. 11. Equipotentials (dashed blue) and field lines (solid red) for geometries described by Eq. (6). (a) An outside corner with $n = 3/2$ and symmetry $*$. (b) A quadrupole lens with $n = 1/2$ and symmetry $*4$.



(a) (b)

FIG. 12. Equipotentials (dashed blue) and field lines (solid red) for (a) the $**$ geometry described by Eq. (7) and (b) the $*22\infty$ geometry described by Eq. (8).

In all cases, the conformal factor is $\Omega = n^2(u^2+v^2)^{n-1}$ and $\partial_u\Omega = 2un^2(n-1)(\Omega/n)^q$, where $q = (n-2)/(n-1)$. Combining these we find

$$\kappa \sim \sigma^\alpha,\ \alpha \equiv \frac{n+1}{n-1}.$$

Any of the equipotentials can be used as a conductor surface. However, the relationships above break down if the conductor is chosen to be the equipotential with the sharp corner, as the curvature is everywhere zero. Note also that for outer corners ($n > 1$), the power $\alpha > 0$, so that curvature is an increasing function of charge, while for inner corners ($n < 1$) the power $\alpha < 0$, so that curvature is a decreasing function of charge.

The second solution is

$$z(w) = \ln\sinh w. \tag{7}$$

This represents a periodic sequence of hills with symmetry type $**$ as shown in Fig. 12. In this case, $\Omega = \coth w \coth\bar{w}$, $\partial_u\Omega = (1-\Omega^2)\tanh 2u$, and the charge-curvature relationship is the increasing function

$$\kappa \sim \frac{\sigma^4 - 1}{\sigma}.$$

The third is

$$z(w) = \arctan\exp w. \tag{8}$$

This solution gives the periodic sequence of oval conductors with symmetry type $*22\infty$ shown in Fig. 12. The conformal factor is $\Omega = (4\cosh w\cosh\bar{w})^{-1}$, the derivative is $\partial_u\Omega = -4\Omega^2\sinh 2u$, and the charge-curvature relationship is the decreasing function

$$\kappa \sim \sigma^{-1}.$$

This example is particularly interesting since the conductors are convex, yet the charge and curvature are inversely related. In particular, σ is greatest where κ is least. This example shows how neighboring conductors can have a greater influence of the charge distribution than the local curvature.

VIII. CONCLUSIONS

We have investigated under what conditions the charge density along the boundary of a planar conductor will be a function of its curvature. Unless the curvature is monotonic, the conductors must transform under one of the planar symmetry groups: when two distinct points on the conductor surface have the same curvature, they will have the same charge density if and only if there is a global symmetry transformation connecting those points. This result highlights the rarity and fragility of a functional relationship, as almost all perturbations of a conductor geometry will destroy the symmetry and with it the functional relationship. While these results have only been obtained for two dimensional conductors, it is natural to conjecture that symmetry will be necessary for obtaining functional relationships in three dimensions.

The explicitly computed examples exhibit a variety of different $\sigma(\kappa)$ relationships, including both increasing and decreasing functions, though for isolated convex conductors only increasing functions were found. It is not known whether there are any restrictions on the possible functions besides monotonicity. In particular, it is an open question whether every strictly monotone function is the $\sigma(\kappa)$ function for some conductor, and if so, whether the conductor geometry can be constructed directly from this function. If a conductor can be defined through a complex map $z(w)$, then it is straightforward to compute Ω and from this $\sigma(\kappa)$ without having to compute either σ or κ as functions of z or w, though these formulas are of limited use if only γ is specified. However, these formulas could be useful for analyzing currents induced by mechanically deforming conductors (particularly in nanoscale devices), especially if the deformation can be described through a family of complex maps. Explicit examples of many of the possible symmetry types, but not all, were found in the Appendix (and shown in Sec. VII). The symmetry of the geometries are implicit in the functional dependence condition on Ω and its normal derivative, but is there a way to make this explicit? Can the symmetry type of a conductor be determined directly from a specific *Ansatze* or functional form of Ω? Finally, can a transformation $z(w)$ be constructed to transform under a specific planar symmetry group?

APPENDIX: SOLUTION TO THE ANSATZE 2 PDE

It is the purpose of this Appendix to solve Eq. (5). First, write $\Omega = f(w)\bar{f}(\bar{w})$, where $f = dz/dw \equiv z'$ and $\bar{f} = d\bar{z}/d\bar{w} \equiv \bar{z}'$. In these variables, Eq. (5) reads

$$f f'\{\bar{f}\bar{f}'''[\bar{f}^2 f'^2 - f^2\bar{f}'^2] + \bar{f}'\bar{f}''[f^2(2\bar{f}\bar{f}'' - \bar{f}'^2) - 3\bar{f}^2 f'^2]\} = [f \leftrightarrow \bar{f}], \tag{A1}$$

where $[f \leftrightarrow \bar{f}]$ indicates the same expression but with f and $\bar{f}$ swapped. As before, a prime denotes differentiation of f with respect to w and of $\bar{f}$ with respect to $\bar{w}$. This equation does not completely separate. It can, however, be simplified using Lie group techniques.[17,18] We first note that Eq. (A1) is invariant under the scaling $f \to \epsilon f$ (and $\bar{f} \to \bar{\epsilon}\bar{f}$). This allows the introduction of the similarity variables $\alpha(s) = f'/f$ and $\bar{\alpha}(s) = \bar{f}'/\bar{f}$, where $s = w$ (the use of the new variable s is not necessary, but is useful to avoid confusion). The functions f and $\bar{f}$ are recovered through integration

$$f = c_0 \exp\int^w \alpha(s)ds \text{ and } \bar{f} = \bar{c}_0 \exp\int^w \bar{\alpha}(s)d\bar{s}, \tag{A2}$$

where c_0 is an arbitrary complex constant.

Using these new variables, Eq. (A1) reduces in degree by one and becomes

$$(\alpha^2 - \bar{\alpha}^2)(\alpha\bar{\alpha}'' - \bar{\alpha}\alpha'') + 2\alpha\bar{\alpha}(\bar{\alpha}'^2 - \alpha'^2) = 0, \tag{A3}$$

where derivatives are with respect to s (and $\bar{s}$). This equation is invariant under translation $s \to s + \epsilon$ (and $\bar{s} \to \bar{s} + \bar{\epsilon}$). This allows the introduction of the new variables $t = \alpha$, $\beta = \alpha'$, and their conjugates. These variables satisfy the first order equation

$$(\bar{t}^2 - t^2)(\bar{t}\beta\beta' + t\bar{\beta}\bar{\beta}') = 2t\bar{t}(\bar{\beta}^2 - \beta^2), \tag{A4}$$

where derivatives are now with respect to t (and $\bar{t}$). Making the *Ansatze* $\beta = \sqrt{p(t)}$ and $\bar{\beta} = \sqrt{\bar{p}(t)}$, transforms Eq. (A4) into

$$(\bar{t}^2 - t^2)(t\bar{p}' + \bar{t}p') = 4t\bar{t}(\bar{p} - p). \tag{A5}$$

Inserting a series solution $p = \sum p_i t^i$, $\bar{p} = \sum \bar{p}_j \bar{t}^j$, we readily find that $p_i = \bar{p}_i$ for $i = 0, 2, 4$ and $p_i, \bar{p}_i = 0$ otherwise, so the solution to Eq. (A4) is

$$\beta(t) = \sqrt{r_1 t^4 + r_2 t^2 + r_3} \text{ and } \bar{\beta}(\bar{t}) = \sqrt{r_1\bar{t}^4 + r_2\bar{t}^2 + r_3}, \tag{A6}$$

where the r_i are *real* constants, so β satisfies the reflection principle: $\beta(\bar{t}) = \bar{\beta}(t)$.

TABLE II. Computation of the integral Eq. (A7) for special values of the constants r_i and solutions for α and f. Here, $\eta = s/C$ and $\xi = w/C$, and C and D are pure real or imaginary constants.

Sol.	r_1	r_2	r_3	$\eta(\alpha)$	$\alpha(\eta)$	$\int^w \alpha$	f/c_0
i.	C^{-2}	0	0	α^{-1}	η^{-1}	$C\ln\xi$	ξ^C
ii.	0	C^{-2}	0	$\ln\alpha$	e^η	e^ξ	$\exp(e^\xi)$
iii.	0	0	C^{-2}	α	η	$C\,\xi^2$	$\exp(C\xi^2)$
iv.	$C^{-2}D^2$	C^{-2}	0	$\ln(\alpha^{-1}+\sqrt{D^2+\alpha^{-2}})$	$2/(e^\eta - D^2e^{-\eta})$	$D\,\mathrm{arctanh}\,e^\xi$	$\coth^{D/2}\xi$
v.	0	C^{-2}	$C^{-2}D^2$	$\ln(\alpha+\sqrt{D^2+\alpha^2})$	$(e^\eta - D^2e^{-\eta})/2$	$e^\xi + D^2e^{-\xi}$	$\exp(e^\xi + D^2e^{-\xi})$
vi.	$C^{-2}D^{-2}$	$2C^{-2}$	$C^{-2}D^2$	$\arctan(\alpha/D)$.	$D\tan\eta$	$D\ln\cos\xi$	$\cos^D\xi$

Since $t = \alpha$ and $ds = d\alpha/\beta$, s is obtained from

$$s(\alpha) = \int^\alpha \frac{dt}{\beta(t)} - c_1 = \int^\alpha \frac{dt}{\sqrt{r_1t^4 + r_2t^2 + r_3}} - c_1, \tag{A7}$$

which must be inverted to obtain α. The integration constant c_1 represents a shift of origin for $s = w = x + iy$, and will henceforth be dropped without loss of generality.

The integral in Eq. (A7) is elliptic, and rather than giving a general formal solution, we enumerate several special cases obtained by suitably choosing the constants to simply the integrand. The first five are obtained by zeroing at least one of the arbitrary constants, while in the sixth, the constants are chosen to make the polynomial a perfect square. For each of these six possibilities, we solve for α in Eq. (A7), integrate, and exponentiate to obtain f. In all cases, the final solution will be a function of $\xi \equiv w/C$, where C is a pure real or pure imaginary constant (C^2 is real) defined in Table II, where all results are tabulated. Note that if C is real, it scales the coordinates, while if it is imaginary, is scales and rotates $w \to iw$, effectively swapping u and v.

Since $f = z'$, the final step is to integrate each of these expressions one last time to obtain $z(w)$. These final integrals are not in the table, but discussed individually below. The final integral for solution (i) is elementary and yields $z = \xi^{1+C}$. For $-1 < C < 1$, this is example 1 in Sec. VII (Eq. (6) and Fig. 11), and for $C = 1$, this is the parabolic coordinates of *Ansatze* 1 in Sec. VII (Fig. 10).

Solution (ii) has the form of an exponential integral $\mathrm{Ei}(1,\xi)$ and describes a periodic solution with $**$ symmetry, qualitatively similar to that defined by solution (iv) below. Solution (iii) integrates to the error function, $c_0\,\mathrm{erf}\,\xi + c_4$, whose coordinates' lines always seem to intersect, and so does not yield a useful solution.

Solution (iv) can be integrated for rational values of D. Interesting solutions are obtained by setting $C = 1/2$ and $D = 2$ yielding $\ln\sinh w$, which is example 2 (Eq. (7) and Fig. 12), or by setting $C = 1/2$ and $D = 1$, yielding $\mathrm{arctanh}\sqrt{\coth w} - \arctan\sqrt{\coth w}$. These all have symmetry type $**$. Solution (v) cannot be solved explicitly. Numerical integration for various values of D reveals equipotentials that always seem to self-intersect, so we do not consider this solution further.

Solution (vi) can be integrated for certain values of C and D. Setting $C = 1$ and $D = 1$ yields $\sin w$, which gives the elliptic/hyperbolic coordinates (*Ansatze* 1 of Sec. VII). Setting $C = i$ and $D = -1$ yields $\arctan\exp w$, which is example 3 (Eq. (8) and Fig. 12). Setting $D = -2$ yields $\tanh w$ and describes a dipole with circular equipotentials, and thus a pair of constant curvature conductors with varying σ (much like the polarized spheres of Refs. 4 and 5), and thus has only the trivial relationship $\kappa(\sigma) = \text{const}$. Other choices of constants would yield additional solutions, though it is unclear whether any would be qualitatively different from those already considered, in particular, whether any would exhibit a distinct symmetry type.

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