Exact Solutions of Five Dimensional Anisotropic Cosmologies

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We solve the five dimensional vacuum Einstein equations for several kinds of anisotropic geometries. We consider metrics in which the spatial slices are characterized as Bianchi types II and V, and the scale factors are dependent both on time and a noncompact fifth coordinate. We examine the behavior of the solutions we find, exploring them within the context of the induced matter model.

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I. INTRODUCTION

Advances in string and membrane theories of unification [1] have boosted interest in higher dimensional cosmologies. An exciting new area of this field involves the realization, in brane world approaches, that extra coordinates need not be compact [2,3]. In the Randall-Sundrum model [4,5], for example, our observed universe, as seen at low-energy scales, is localized to four dimensions by a 3-brane embedded within a five dimensional bulk. An interesting question concerns the range of geometries that would support such localization. These are characterized by a warp factor, dependent in general on the fifth coordinate, that determines the geometry of the 4D part of the metric [6,7]. The 5D Einstein equations, along with boundary conditions, set the form of this warp factor.

Although workers in this field have focused mainly on isotropic geometries, the 5D Einstein equations admit a much wider range of solutions, including those with anisotropic dynamics. Recently, Frolov has obtained a class of exact anisotropic solutions within the context of the brane world approach, in which the behavior of the bulk resembles the Kasner solution [8].

The Kasner solution represents the simplest type of anisotropic behavior. The range of possible anisotropic geometries can be characterized by Bianchi's classification, first applied to cosmology by Taub in 1951 [9]. Calculating the field equations for each Bianchi type, Taub examined spatially homogeneous, vacuum-filled space times, classifying the Kasner solution as Bianchi type I and finding a new exact solution in the case of type II.

To extend these results to higher dimensions and understand the range of possible behaviors, we have examined several different anisotropic five dimensional cosmologies, including generalized versions of type II and type V, comparing these solutions to our previous results for type I [10]. We examine these within the context of the induced matter approach, proposed by Wesson [11], and shown to have a close connection to brane world theories [12]. The induced matter approach, for which many exact solutions have been found [13–19] postulates that the additional geometrical terms arising from a noncompact fifth coordinate in the five-dimensional vacuum Einstein equations can be associated with the matter-energy components of four-dimensional theory. In other words, the dynamics of the observed 4D matter- and energy-filled universe are equivalent to that of a 5D empty Kaluza-Klein theory—rendering material from pure geometry. Though the induced matter approach includes a non-compact fifth coordinate, in contrast to brane world models, it is a classical 5D vacuum theory. In this spirit, we look at 5D vacuum models of type II and type V, examining their 4D implications. Ultimately, our aim is to help answer the following question: of the range of solutions to the five dimensional Einstein equations (including the full variety of anisotropic geometries), which of these exhibit long term behavior consistent with observed present-day conditions?

II. 5D BIANCHI TYPE-II SOLUTION

We consider the five dimensional metric

$$ds^2 = e^{\nu} dt^2 - g_{ij}\omega^i\omega^j - e^{\mu} dl^2 \tag{2.1}$$

where the 3D spatial part of the metric can be expressed in diagonal form as

$$g_{ij} = \mathrm{diag}(e^{\alpha}, e^{\beta}, e^{\gamma}). \tag{2.2}$$

The time coordinate t and the three spatial coordinates x, y and z have been supplemented with a fifth coordinate l. We assume that the metric coefficients μ, ν, α, β and γ each depend, in general, on both t and l.

The one-forms ω^i have the relationship

$$d\omega^i = \frac{1}{2} C^i{}_{jk}\omega^j\omega^k, \tag{2.3}$$

where the $C^i{}_{jk}$ are the structure constants corresponding to the particular Bianchi type.

In the case of type II, the non-zero structure constants are

$$C^1{}_{23} = -C^1{}_{32} = 1. \tag{2.4}$$

Solving the 5D Einstein equations for the vacuum case, without a cosmological term, we find two distinct sets of solutions. The first class of solutions is independent of l and represents an extension of Taub's 4D solution. It can be stated as

$$\alpha = 2a_1 t - \ln[\cosh(t+t_0)] \tag{2.5}$$

$$\beta = 2b_1 t + \ln[\cosh(t+t_0)] \tag{2.6}$$

$$\gamma = 2c_1 t + \ln[\cosh(t+t_0)] \tag{2.7}$$

$$\mu = -4a_1 t \tag{2.8}$$

$$\nu = -4a_1 t + 4b_1 t + 2\ln[\cosh(t+t_0)] \tag{2.9}$$

where

$$c_1 = \frac{8a_1{}^2 + 4a_1 b_1 + 1}{4b_1 - 4a_1}. \tag{2.10}$$

Following Wesson's procedure we can rewrite the 5D vacuum Einstein equations, $G^{\mu}{}_{\nu}=0$, as 4D equations with induced matter by collecting each of the terms in G_0^0, G_1^1, G_2^2 and G_3^3 dependent on either μ or on derivatives with respect to l, placing these quantities on the right-hand side of the equations and identifying them respectively as the induced matter density and pressure components:

$$8\pi\rho = e^{-\nu}\left(-\frac{1}{4}\alpha^{\cdot}\mu^{\cdot} - \frac{1}{4}\beta^{\cdot}\mu^{\cdot} - \frac{1}{4}\gamma^{\cdot}\mu^{\cdot}\right) + e^{-\mu}\left(\frac{1}{2}\alpha^{**} + \frac{1}{2}\beta^{**} + \frac{1}{2}\gamma^{**} + \frac{1}{4}\alpha^{*2} + \frac{1}{4}\beta^{*2} + \frac{1}{4}\gamma^{*2} - \frac{1}{4}\mu^{*}\alpha^{*} - \frac{1}{4}\mu^{*}\beta^{*} - \frac{1}{4}\mu^{*}\gamma^{*} + \frac{1}{4}\alpha^{*}\beta^{*} + \frac{1}{4}\alpha^{*}\gamma^{*} + \frac{1}{4}\beta^{*}\gamma^{*}\right) \tag{2.11}$$

$$8\pi p_1 = e^{-\nu}\left(\frac{1}{2}\mu^{\cdot\cdot} + \frac{1}{4}\mu^{\cdot 2} + \frac{1}{4}\beta^{\cdot}\mu^{\cdot} + \frac{1}{4}\gamma^{\cdot}\mu^{\cdot} - \frac{1}{4}\mu^{\cdot}\nu^{\cdot}\right) + e^{-\mu}\left(-\frac{1}{2}\beta^{**} - \frac{1}{2}\gamma^{**} - \frac{1}{2}\nu^{**} - \frac{1}{4}\beta^{*2} - \frac{1}{4}\gamma^{*2} - \frac{1}{4}\nu^{*2} - \frac{1}{4}\beta^{*}\gamma^{*} - \frac{1}{4}\beta^{*}\nu^{*} - \frac{1}{4}\gamma^{*}\nu^{*} + \frac{1}{4}\beta^{*}\mu^{*} + \frac{1}{4}\gamma^{*}\mu^{*} + \frac{1}{4}\mu^{*}\nu^{*}\right) \tag{2.12}$$

$$8\pi p_2 = e^{-\nu}\left(\frac{1}{2}\mu^{\cdot\cdot} + \frac{1}{4}\mu^{\cdot 2} + \frac{1}{4}\alpha^{\cdot}\mu^{\cdot} + \frac{1}{4}\gamma^{\cdot}\mu^{\cdot} - \frac{1}{4}\mu^{\cdot}\nu^{\cdot}\right) + e^{-\mu}\left(-\frac{1}{2}\alpha^{**} - \frac{1}{2}\gamma^{**} - \frac{1}{2}\nu^{**} - \frac{1}{4}\alpha^{*2} - \frac{1}{4}\gamma^{*2} - \frac{1}{4}\nu^{*2} - \frac{1}{4}\alpha^{*}\gamma^{*} - \frac{1}{4}\alpha^{*}\nu^{*} - \frac{1}{4}\gamma^{*}\nu^{*} + \frac{1}{4}\alpha^{*}\mu^{*} + \frac{1}{4}\gamma^{*}\mu^{*} + \frac{1}{4}\mu^{*}\nu^{*}\right) \tag{2.13}$$

$$8\pi p_3 = e^{-\nu}\left(\frac{1}{2}\mu^{\cdot\cdot} + \frac{1}{4}\mu^{\cdot 2} + \frac{1}{4}\alpha^{\cdot}\mu^{\cdot} + \frac{1}{4}\beta^{\cdot}\mu^{\cdot} - \frac{1}{4}\mu^{\cdot}\nu^{\cdot}\right) + e^{-\mu}\left(-\frac{1}{2}\alpha^{**} - \frac{1}{2}\beta^{**} - \frac{1}{2}\nu^{**} - \frac{1}{4}\alpha^{*2} - \frac{1}{4}\beta^{*2} - \frac{1}{4}\nu^{*2} - \frac{1}{4}\alpha^{*}\beta^{*} - \frac{1}{4}\alpha^{*}\nu^{*} - \frac{1}{4}\beta^{*}\nu^{*} + \frac{1}{4}\alpha^{*}\mu^{*} + \frac{1}{4}\beta^{*}\mu^{*} + \frac{1}{4}\mu^{*}\nu^{*}\right) \tag{2.14}$$

where we use overdots to represent partial derivatives with respect to t, and asterisks to represent partial derivatives with respect to l.

Substituting our solution into these components, we obtain

$$8\pi\rho = a_1\left[\tanh(t+t_0) + \frac{(4a_1{}^2 + 4a_1 b_1 + 4b_1{}^2 + 1)}{2(b_1 - a_1)}\right]e^{f(t)} \tag{2.15}$$

$$8\pi p_1 = a_1\left[\frac{(8a_1{}^2 + 4b_1{}^2 + 1)}{2(b_1 - a_1)}\right]e^{f(t)} \tag{2.16}$$

$$8\pi p_2 = a_1\left[\tanh(t+t_0) + \frac{(12a_1{}^2 - 8a_1 b_1 + 8b_1{}^2 + 1)}{2(b_1 - a_1)}\right]e^{f(t)} \tag{2.17}$$

$$8\pi p_3 = a_1\left[2\tanh(t+t_0) + \frac{(10a_1{}^2 + 2b_1{}^2 + 1)}{(b_1 - a_1)}\right]e^{f(t)} \tag{2.18}$$

where

$$f(t) = \frac{(12a_1{}^2 - 4a_1 b_1 + 4b_1{}^2 + 1)t}{(a_1 - b_1)} - 2\ln[\cosh(t+t_0)]. \tag{2.19}$$

To ensure that the density of the induced matter is strictly positive, we find that $a_1 > 0$ and $b_1 > a_1$. This first condition mandates that $\mu < 0$ for all t, forcing the scale factor associated with the fifth coordinate to be a monotonically decreasing function. Thus the positive density requirement naturally leads to dimensional reduction, similar to the type first described by Chodos and Detweiler for 5D Kasner solutions [20]. This range of values for a_1 and b_1 also ensures that c_1 is positive, guaranteeing that the three spatial scale factors expand over time. Note, however, in contrast to many other non-compact Kaluza-Klein approaches (such as Randall-Sundrum), the scale factors of this solution lack a "warp factor" dependency on the fifth coordinate, and would not reproduce standard observed 4D gravity in this form.

III. A CLASS OF 5D BIANCHI TYPE-II SOLUTIONS WITH EXPONENTIAL BEHAVIOR

Further investigating the 5D Einstein equations we find a second set of five dimensional solutions with type-II geometry. In contrast with the first set, this class of solutions is dependent on both t and l. The metric coefficients can be expressed in the following manner:

$$\alpha = 2a_1 t + 2a_2 l \tag{3.1}$$

$$\beta = 2b_1 t + 2b_2 l \tag{3.2}$$

$$\gamma = 2c_1 t + 2c_2 l \tag{3.3}$$

$$\mu = \nu = 2d_1 t + 2d_2 l \tag{3.4}$$

where

$$a_1 = \frac{-2a_2{}^2 + 2a_2c_2 + 1 + a_2\sqrt{6}\sqrt{2a_2{}^2 + 2c_2{}^2 - 1}}{2\sqrt{4a_2{}^2 - 1 - 2a_2c_2 + 4c_2{}^2 + \sqrt{6}\sqrt{(2a_2{}^2 + 2c_2{}^2 - 1)(a_2 - c_2)^2}}} \tag{3.5}$$

$$b_1 = \frac{-6a_2{}^2 + 4a_2c_2 - 10c_2{}^2 + 2 + (a_2 - 3c_2)\sqrt{6}\sqrt{2a_2{}^2 + 2c_2{}^2 - 1}}{2\sqrt{4a_2{}^2 - 1 - 2a_2c_2 + 4c_2{}^2 + \sqrt{6}\sqrt{(2a_2{}^2 + 2c_2{}^2 - 1)(a_2 - c_2)^2}}} \tag{3.6}$$

$$c_1 = \frac{-2a_2c_2 - 1 + 2c_2{}^2 + c_2\sqrt{6}\sqrt{2a_2{}^2 + 2c_2{}^2 - 1}}{2\sqrt{4a_2{}^2 - 1 - 2a_2c_2 + 4c_2{}^2 + \sqrt{6}\sqrt{(2a_2{}^2 + 2c_2{}^2 - 1)(a_2 - c_2)^2}}} \tag{3.7}$$

$$d_1 = -\frac{2a_2{}^2 + 4c_2{}^2 + c_2\sqrt{6}\sqrt{2a2^2 + 2c_2{}^2 - 1}}{\sqrt{4a_2{}^2 - 1 - 2a_2c_2 + 4c_2{}^2 + \sqrt{6}\sqrt{(2a2^2 + 2c_2{}^2 - 1)(a_2 - c_2)^2}}} \tag{3.8}$$

$$b_2 = -2c_2 - \frac{\sqrt{6}}{2}\sqrt{2a_2{}^2 + 2c_2{}^2 - 1} \tag{3.9}$$

$$d_2 = -a_2 - c_2 - \frac{\sqrt{6}}{2}\sqrt{2a_2{}^2 + 2c_2{}^2 - 1} \tag{3.10}$$

and a_2 and c_2 are independent parameters with $c_2 > a_2$.

Interestingly, this set of solutions is purely exponential in character, with monotonic behavior similar to the 5D Kasner (type-I) solution explored in our earlier study. Note, however, that the relationship amongst these exponents is more complex than in the Kasner case. Also, unlike the 5D Kasner solution, in this case it is impossible for three of the scale factors to behave identically. Therefore this type-II model cannot expand isotropically in three directions while contracting in the fourth.

For this class of solutions the density and pressure of the induced matter can be calculated to be

$$8\pi\rho = [-2c_2{}^2 - a_2c_2 - 1/2(a_2 + c_2)\sqrt{6}\sqrt{2a_2{}^2 + 2c_2{}^2 - 1}]e^{\kappa_1 t + \kappa_2 l} \tag{3.11}$$

$$\begin{aligned} 8\pi p_1 = {} & [-14c_2{}^4 + 20a_2c_2{}^3 + 13c_2{}^2 - 22c_2{}^2a_2{}^2 - 9c_2a_2 + 16c_2a_2{}^3 - 3/2 - 6a_2{}^4 + 8a_2{}^2 \\ & + (5/2c_2 - 3/2a_2 - 4c_2{}^3 - 4a_2{}^2c_2 + 6c_2{}^2a_2 + 2a_2{}^3)\sqrt{6}\sqrt{2a_2{}^2 + 2c_2{}^2 - 1}]e^{\kappa_1 t + \kappa_2 l} \end{aligned} \tag{3.12}$$

$$\begin{aligned} 8\pi p_2 = {} & [-14c_2{}^4 - 8a_2c_2{}^3 - 10c_2{}^2a_2{}^2 + 4c_2{}^2 - 8c_2a_2{}^3 + 4c_2a_2 - 2a_2{}^4 + a_2{}^2 \\ & - (4c_2{}^3 + 2a_2c_2{}^2)\sqrt{6}\sqrt{2a_2{}^2 + 2c_2{}^2 - 1}]e^{\kappa_1 t + \kappa_2 l} \end{aligned} \tag{3.13}$$

$$\begin{aligned} 8\pi p_3 = {} & [-14a_2{}^4 + 28c_2a_2{}^3 - 34c_2{}^2a_2{}^2 + 10a_2{}^2 + 28a_2c_2{}^3 - 14c_2a_2 - 14c_2{}^4 + 10c_2{}^2 - 3/2 \\ & - (4a_2{}^2 - 2a_2c_2 + 2c_2{}^2 - 1)\sqrt{6}\sqrt{(2a_2{}^2 + 2c_2{}^2 - 1)(c_2 - a_2)^2}]e^{\kappa_1 t + \kappa_2 l}/[4a_2{}^2 - 2a_2c_2 \\ & + 4c_2{}^2 - 1 + \sqrt{6}\sqrt{2a_2{}^2 + 2c_2{}^2 - 1}] \end{aligned} \tag{3.14}$$

where

$$\kappa_1 = \frac{8c_2{}^2+4a_2{}^2+2c_2\sqrt{6}\sqrt{2a_2{}^2+2c_2{}^2-1}}{\sqrt{4c_2{}^2-2a_2c_2+4a_2{}^2-1+\sqrt{6}\sqrt{(2a_2{}^2+2c_2{}^2-1)(c_2-a_2)^2}}} \tag{3.15}$$

$$\kappa_2 = 2(a_2+c_2)+\sqrt{6}\sqrt{2a_2{}^2+2c_2{}^2-1}. \tag{3.16}$$

To guarantee positive induced matter density, one can choose $c_2 < 0$.

IV. 5D BIANCHI TYPE-V SOLUTION

We now consider 5D models with Bianchi type-V spatial geometry. We write the metric in the same manner as Eq. (2.1) with the non-zero structure constants of the Lie algebra of the one-forms equal to

$$C^1{}_{13} = -C^1{}_{31} = 1 \tag{4.1}$$

$$C^2{}_{23} = -C^2{}_{32} = 1. \tag{4.2}$$

The 5D Einstein equations yield the following solution:

$$\alpha = 2a_1 t + \sqrt{2} a_2 l \tag{4.3}$$

$$\beta = 2\ \ln(1/2\sqrt{-4a_1{}^2+2a_2{}^2}) - 2a_1 t - \sqrt{2} a_2 l \tag{4.4}$$

$$\gamma = 2,a_2 t - 2\ \ln(1/2\sqrt{-4a_1{}^2+2a_2{}^2}) + 2a_1\sqrt{2} l \tag{4.5}$$

$$\mu = \nu = 2a_2 t + 2a_1\sqrt{2} l \tag{4.6}$$

where a_1 and a_2 are independent parameters with $a_2{}^2 > 2a_1{}^2$ to ensure that all scale factors are real.

Substituting this solution into equations, the density and pressure of the induced matter reduce to

$$8\pi\rho = 8\pi P_3 = -1/2(a_2{}^2) e^{-2a_2 t - 2a_1\sqrt{2} l} \tag{4.7}$$

$$8\pi P_1 = 8\pi P_2 = 1/2(a_2{}^2 - 4a_1{}^2) e^{-2a_2 t - 2a_1\sqrt{2} l}. \tag{4.8}$$

Note, however, that $\rho \leqslant 0$ for all values of the parameters, violating the usual supposition of positive density.

V. CONCLUSIONS

We have found exact solutions for several types of five dimensional anisotropic vacuum cosmologies, representing generalizations of the Bianchi type-II and type-V models. One class of generalized Bianchi type-II cosmologies resembles a higher dimensional extension of Taub's solution. It exhibits natural cosmological dimensional reduction, generated by the assumption of positive induced matter density. Another class of generalized type-II models more closely resembles the monotonic behavior of the Kasner model, albeit with a more complex relationship among its parameters. Finally we have found a class of five dimensional cosmologies with type-V spatial geometries, also exhibiting monotonic exponential behavior, dependent on both the fifth coordinate and time. To extend this preliminary study, future work will examine the effects of a negative cosmological term, considering anisotropic generalizations of five dimensional anti−de Sitter geometries.

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