The Lorentz force and the radiation pressure of light

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I. THE FRESHMAN ARGUMENT

The interaction of light and matter plays a central role, not only in physics itself, but in any introductory electricity and magnetism course. To develop this topic, most courses introduce the Lorentz force law, which gives the electromagnetic force acting on a charged particle, and later discuss Maxwell’s equations. Students are then persuaded that Maxwell’s equations admit wave solutions that travel at the speed of light, thus establishing the connection between light and electromagnetic waves. At this point instructors generally state that electromagnetic waves carry momentum in the direction of propagation. Tipler and Mosca, for example, then derive an expression for the radiation pressure produced by a light wave. A cursory look at their argument shows that the “freshman” argument is incorrect, to show that nontrivial derivation of radiation pressure that is accessible to first-year students. Consider, then, the situation mathematically incorrect and has several serious conceptual difficulties without obvious resolution at the classical, yet alone introductory, level. We discuss these difficulties and propose an alternative argument. © 2009 American Association of Physics Teachers.

II. EQUATION OF MOTION

To determine the momentum of the charge, which we take to be an electron, assume the electric and magnetic fields of the light wave are given by \( E = E_0 \sin(\omega t + \phi) \hat{i} \) and \( B = B_0 \sin(\omega t + \phi) \hat{j} \), where \( \phi \) is an arbitrary phase angle. In our units \( E_0 = B_0 \). We set \( \vec{F}_{\text{Lorentz}} = m \vec{v} / \text{dt} \) in Eq. (2) to obtain a pair of coupled ordinary linear first-order equations for the electron velocity.
eral graphs for various values of \( \phi \), but can easily be plotted. In Figs. 2–4 we show several graphs for various values of \( \omega_c/\omega \) and phase angle \( \phi \).

Note that regardless of \( \phi \), \( v_z \) is always positive, but that there is also a nonzero \( v_x \) whose average can be positive, negative, or zero depending on \( \phi \). The \( \phi=0 \) case is shown in Figs. 2 and 4 and the \( \phi=\pi/2 \) case in Fig. 3. Also, for \( v_z \ll 1, v_x \gg v_z \).

Additional insight into the solutions can be obtained by considering the limit \( \omega_c/\omega \ll 1 \). For ordinary light sources at optical frequencies \( \omega \sim 10^{16} \text{ rad/s} \), consideration of the Poynting flux (in the following section) gives \( \omega_c/\omega \sim 10^{-11} \), and so the limit is well satisfied. For high-powered lasers, such as those at the National Ignition Facility with pulse energy \( \sim 2 \text{ MJ} \), it is possible that \( \omega_c/\omega \) exceeds \( \omega \). For \( \omega_c \ll \omega \), expansion of the solutions (6) to lowest order in \( \omega_c/\omega \) for \( \phi=0 \) yields

\[
\begin{align*}
v_z &\equiv \frac{1}{2} \left( \frac{\omega_c}{\omega} \right)^2 \left[ \cos(\omega t) - 1 \right]^2, \\
v_x &\equiv \left( \frac{\omega_c}{\omega} \right) \left[ 1 - \cos(\omega t) \right].
\end{align*}
\]

Both \( v_x \) and \( v_z \) are positive definite, as shown in Fig. 2. Therefore their averages must be as well. This in itself contradicts the arguments of Ref. 2 that \( \langle F_z \rangle = 0 \) but that \( \langle F_z \rangle \neq 0 \). Note also that \( v_z \) is of order \( (\omega_c/\omega)^2 \), and \( v_x \) is of order \( \omega_c/\omega \). The behavior coincides with the plots, but suggests that because \( v_x^2 \sim (\omega_c/\omega)^2 \), a consistent, relativistic calculation will significantly change \( v_z \). Moreover, the time averages of both \( v_z \) and \( v_x \) vanish to all orders, and so it is in fact impossible to exert a net force on the particle.

One might object to the arguments of this section on the grounds that we have taken \( E \) and \( B \) to be simple harmonic \( \sim \sin(\omega t) \) rather than wave-like \( \sim \sin(kz - \omega t) \). However, it is evident from Eq. (7) that \( kz \ll \omega t \) always and that such corrections are therefore negligible, an assertion borne out by numerical calculations.
III. INTERPRETATION

The question is whether the behavior just discussed can be reconciled with the classical picture of the Poynting flux. The Poynting vector in our units is

$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi}, \quad (8)$$

and the time average is \( \langle \mathbf{S} \rangle = \text{Re}(\mathbf{E} \times \mathbf{B}^*)/4\pi \). \( \mathbf{S} \) points in the direction of propagation of the electromagnetic wave and in units with \( c = 1 \) can be regarded interchangeably as power per unit area, energy per unit volume (or pressure), or momentum flux. If the freshman argument is correct, then the particle should be accelerated in the direction of the Poynting vector. But our previous results show that, on the contrary, the particle drifts off in some other direction at a constant average velocity.

Unfortunately, there seem to be several deep inconsistencies in the entire approach. One is that the freshman argument is an invalid attempt to apply the standard classical derivation that is invoked to identify the Poynting flux with the Lorentz force on a volume of charges: the electron must gain the same amount. Multiplying by the momentum flux of photons will give the total force on the electron. Because the Poynting flux is the momentum flux of photons, the same numerical result is obtained by multiplying the Thomson cross section by the momentum component; the electron must gain the same amount. Multiplying the differential Thomson scattering cross section in Eq. (11) by \( 1-\cos \theta \) and integrating over the sphere gives the total Thomson scattering cross section

$$\sigma_T = \frac{8\pi}{3} \left( \frac{e^2}{m} \right)^2. \quad (12)$$

Multiplication by the momentum flux of photons will give the total force on the electron. Because the Poynting flux is the momentum flux of photons, the same numerical result is obtained by multiplying the Thomson cross section by the time-averaged Poynting flux. This argument, however, relies on the quantum nature of photons. The Thomson cross section is the nonrelativistic limit of a cross section that must ultimately be derived from QED, and so we see that the freshman argument leads quickly to a situation that might lead quickly into quantum territory.

A second difficulty is that the assumption of plane waves with constant amplitude is an assumption of constant energy and momentum. If the light wave has constant momentum, how can any be transferred to the electron? There are many instances in physics where we ignore the backreaction of a recoiling particle on the system. For instance, according to conservation of momentum, a ball should not bounce off a wall, until it is realized that the ball’s change in momentum is absorbed by Earth.

Holding the amplitude constant in the current calculation might seem a reasonable approximation, but to be totally consistent we should take into consideration the fact that the electron is accelerating and consequently emits radiation, and with that radiation momentum. The customary way to do this calculation in the nonrelativistic limit is via Thomson scattering. The differential Thomson scattering cross section for a wave polarized in the \( x \)-direction is

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} \left( \frac{e^2}{m} \right)^2 \left[ \cos^2 \theta \cos^2 \phi + \sin^2 \phi \right], \quad (11)$$

where \( \theta \) is the angle between the incident and scattered wave. The differential scattering cross section is defined as the ratio of the radiated power per unit solid angle to the incident power per unit area.

Fig. 4. The same as Fig. 2 except that \( \omega_1/\omega_0 = 0.5 \).
have no resolution in the realm of classical physics!

The failing of Thomson scattering is due to the fact that no energy is removed from the original beam. A possible classical “out” to this situation is to assert that the energy radiated by the electron must be that lost by the incoming beam. Therefore because \( E=cp \) for a classical wave, momentum conservation implies that the electron must acquire a \( z \)-momentum as for the Compton scattering case just discussed.\(^7\) Although this argument is valid in terms of conservation laws, it gives no mechanism for transferring the energy from the incident wave to the electron. Unfortunately, modeling the process as interference between the incident plane wave and the spherical wave outgoing from the electron fails to result in any transfer of \( z \)-momentum from the wave to the charge. To recover the Compton result eventually requires including the radiation-reaction force on the electron, which we now consider, but because this derivation involves the classical radius of the electron, it has already gone beyond the realm of classical electromagnetism.

The most straightforward way to deal with the failure of the classical approaches is via the Abraham-Lorentz model, which accounts for the energy radiated by the electron, if in a somewhat \textit{ad hoc} manner. From the Larmor formula the energy radiated by an accelerated electron over a time \( T \) is \( \sim 2e^2d^2T/3 \). Equating this energy to the kinetic energy lost by the particle \( \sim ma^2T^2 \) gives a characteristic time to lose all the energy to radiation:

\[
\tau = \frac{2e^2}{3m}. \tag{13}
\]

This timescale is \( 2/3 \) the time for light to cross the classical radius of the electron, \( r_c = e^2/m \), and has a value \( \tau \sim 10^{-23} \text{s} \). The total force acting on a particle is now \( mv = F_{\text{ext}} + F_{\text{rad}} \), where \( F_{\text{rad}} \) is the radiation-reaction force. Conservation of energy considerations led Abraham and Lorentz to propose that \( F_{\text{rad}} = m\mathbf{v} \) (see Ref. \( 7 \) for more details) and consequently obtain the famous formula

\[
m(\mathbf{v} - \mathbf{r}) = F_{\text{ext}}. \tag{14}\]

With sufficient massaging, Eq. \( (14) \) can be applied to the present circumstance to obtain the desired answer, that is, the force imparted to the electron by an electromagnetic wave is \( F = \langle S \rangle \sigma_T \). Equation \( (2) \) now becomes

\[
\dot{\mathbf{v}}_\perp = \frac{e}{m}(E_x - v_yB_z), \tag{15a}
\]

\[
\dot{\mathbf{v}}_z = \frac{e}{m}v_yB_y. \tag{15b}
\]

In the nonrelativistic regime \( v_x \ll 1 \) and we ignore the second term on the right in Eq. \( (15a) \). We also take both \( \mathbf{v}_x \) and \( \mathbf{v}_z \) to be of the form \( \mathbf{v} = v_0e^{-i\omega t} \), which is of course manifestly untrue according to the results of Sec. II. Then \( \dot{\mathbf{v}}_x = -i\omega v_x \) and \( \dot{\mathbf{v}}_z = -i\omega v_z \). Equation \( (15a) \) becomes

\[
-i\omega v_x (1 + i\omega t) \equiv \frac{e}{m}E_x, \tag{16}\]

or with \( \omega \tau \ll 1 \)

\[
v_x \equiv \frac{ie}{m\omega}E_x(1 - i\omega t). \tag{17}\]

With the assumption that \( \omega / \omega_0 \ll 1 \) and \( \omega \tau \ll 1 \) the \( \dot{\mathbf{v}}_z \) term in Eq. \( (15b) \) can be ignored. Then

\[
\dot{\mathbf{v}}_z \equiv \frac{ie^2}{m^2\omega}E_xB_y(1 - i\omega t). \tag{18}\]

For simplicity, take \( E_x \) and \( B_y \) to be real. We want the time average of the real part of this expression, or

\[
\langle F_z \rangle = \frac{\epsilon^4}{m^2}E_xB_y(1 - \frac{1}{3}) = \langle S \rangle \sigma_T. \tag{19}\]

The earliest paper we have found that proposes this calculation is by Page in 1920,\(^6\) although one suspects that Eddington carried it out earlier. Clearly there are a few things to be desired in the derivation, but it does show that the radiation-reaction force is necessary to obtain the claimed result.

With slightly more work the conclusion can be put on a firmer footing via a perturbation calculation\(^9\) as follows. Note that Eq. \( (7) \) is the zeroth-order solution of Eq. \( (15) \), that is, when \( \tau = 0 \) and \( v_x \ll 1 \) is neglected. Assume \( \mathbf{v}_x = v_0 e^{i\omega t} + v_x \) and \( \mathbf{v}_z = v_z \), where the subscript \( 0 \) refers to the zeroth-order solution and the subscript \( 1 \) refers to the perturbation. It is then not too difficult to show that the surviving \( \mathbf{v}_z \) is the perturbative part:

\[
\mathbf{v}_{z,1} \equiv \epsilon_0 v_z \sin(\omega t) \equiv \epsilon_0^2 \tau \sin^2(\omega t). \tag{20}\]

Taking the time average of this expression vindicates the previous result. We emphasize that the Abraham-Lorentz model includes an explicit statement about the structure of the electron and hence cannot be regarded as entirely classical; the model is a transition to quantum mechanics and quantum field theory.

### IV. ALTERNATIVE APPROACH

Despite the many pitfalls revealed by the above methods, there is a superior and convincing demonstration that light exerts a pressure on matter, one that should be accessible to students who have had a basic exposure to Maxwell’s equations. The advantage of this demonstration is that it avoids consideration of the force acting on a point charge and can therefore be carried out at the purely classical level. For this reason it should be adopted by introductory textbook authors. What follows is a simplified version of a calculation described by Planck.\(^10\)

As before, consider a light wave propagating in the \( +z \)-direction that bounces off a mirror at \( z = 0 \) (see Fig. 5). We take the mirror to be a near perfect conductor of height \( dx \), width \( dy \), and thickness \( z \). The electric field of the light is a superposition of right- and left-traveling waves:

\[
E_x = E_0 \cos(kz - \omega t) - E_0 \cos(kz + \omega t), \tag{21}\]

where \( k = 2\pi/l \) is the wave number, and we have included a phase change on reflection. (This solution ensures that \( E = 0 \) at the surface of the conductor. Recall that the tangential component of an \( E \)-field must be continuous across a boundary, and because the interior field essentially vanishes for a good conductor, the exterior field at the boundary must also.)

From the differential form of Faraday’s law,\(^11\)

\[
\nabla \times \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t}, \text{ we have}
\]

\[
\nabla \times \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t}, \text{ we have}
\]
Integrating with respect to \( t \) and remembering that \( k = \omega \) in units where \( c = 1 \) gives

\[
B = E_0 \left[ \cos(kz - \omega t) + \cos(kz + \omega t) \right] j = 2B_0 \cos(kz) \cos(\omega t) j. \tag{23}
\]

Notice that at the boundary, \( B = 2B_0 \cos(\omega t) \neq 0 \) and that therefore by Ampère’s law, \( \oint \mathbf{B} \cdot d\mathbf{s} = 4\pi I \), oscillating currents must be induced near the surface of the mirror. Because \( B \) is in the \( \pm y \)-direction, the right-hand-rule tells us that these currents will be in the \( \pm x \)-direction, and that \( \mathbf{I} \times \mathbf{B} \) will always point in the \( +z \)-direction. Consequently, the Lorentz force due to the light, \( \mathbf{F} = I d\mathbf{x} \times \mathbf{B} \) for a mirror of height \( dx \) and total current \( I \), will produce a force in the direction of propagation.

We can calculate the magnitude of the force simply and plausibly. The magnitude of the Lorentz force is \( dF = I dx B_z \), or \( dF = I dx dy dz B \) for current density \( J \). The differential form of Ampère’s law tells us that

\[
\nabla \times \mathbf{B} = -\frac{\partial \mathbf{E}}{\partial t} = 4\pi \mathbf{J}, \tag{24}
\]

or \( J = -(1/4\pi) \mathbf{E} \cdot \mathbf{B} / \partial z \). The Lorentz force therefore becomes

\[
\frac{dF}{dx dy dz} = -\frac{1}{4\pi} \frac{\partial B_z}{\partial z} B_0 dz. \tag{25}
\]

The quantity on the left is \( dP \), where \( P \) is the pressure. Because the only spatial dependence of \( B \) is on \( z \), we can ignore the distinction between the partial and full differentials. Evidently, because \( \partial B_z / \partial z \) is connected to \( J \), we must interpret \( B \) as being the field exerting a force on a given slice within the conductor. If we assume that the magnetic field drops off to zero at infinity, which is certainly true inside a good conductor where the falloff is exponential, the total pressure on the mirror should be

\[
P = -\frac{1}{4\pi} \int_0^\infty B \ dB = \frac{1}{8\pi} B(0)^2 = \frac{1}{2\pi} B_0^2 \cos^2(\omega t), \tag{26}
\]

where the last equality follows from Eq. (23) and the continuity of the tangential component of \( \mathbf{B} \) across the boundary. The time average of Eq. (26) gives

\[
P = \frac{E_0 B_0}{4\pi} = 2(S)_{\text{incident}} \tag{27}
\]

as desired. Note that the factor of 2 is expected due to the recoil of the wave off the mirror.

There are a few tacit assumptions in this derivation that should be made explicit. One might wonder, for example, why we used Ampère’s law (24) to calculate the conduction current, rather than Faraday’s law, \( d\phi/dt = -\oint \mathbf{E} \cdot d\mathbf{s} = \mathcal{E} \), for the magnetic flux \( \phi = B dx dz \) and the induced EMF \( \mathcal{E} \). Normally, we would have students use this law to calculate the induced current \( I = \mathcal{E} / R \) in, for example, a wire loop of resistance \( R \). However, in a good conductor \( E \ll B \) and hence \( |d\phi/dt| = \oint \mathbf{E} \cdot d\mathbf{s} = \oint \mathbf{B} \cdot d\mathbf{s} = 4\pi I \), the last equality representing Ampère’s law.

Furthermore, the \( B \)-field in Eq. (24) includes both the incident field and that generated by the induced currents. It seems unreasonable that the portion of the \( B \)-field generated by the induced currents can result in a net force on the currents themselves (no “Munchausen effect”\(^{12}\)). A detailed calculation demonstrates that the integrated force exerted on the induced currents by the induced \( B \)-field vanishes. With these assumptions the simpler derivation we have presented is sound and shows that light waves do exert a pressure on matter in the direction of propagation.

In conclusion, although one does not, and cannot, expect derivations at the introductory level to be uniformly rigorous, this case is of particular interest because the interaction of light with matter is of fundamental importance. Moreover, the explanation presented in some textbooks is so seriously flawed that even students sometimes notice the difficulties. Rather than try to paper over these problems with what must be regarded as nonsensical arguments, the occasion would be better exploited to point out that physics is composed of a collection of models that are brought to bear in explaining physical phenomena, but that these models have limited domains of applicability and as often as not are inconsistent.

ACKNOWLEDGMENT

The authors would like to thank Jim Peebles for suggesting the mirror argument.

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\(^{12}\)Reference 2, p. 363.


1975).

7David Bohm, Quantum Theory (Prentice Hall, New York, 1951), p. 34.


9Reference 7, p. 34.


11Most introductory texts use the integral form of Maxwell’s equations.

The derivation can easily be carried out by considering infinitesimal loops in the \(xz\) and \(yz\) planes as follows: The integral form of Faraday’s law is \(\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi}{dt}\) for magnetic flux \(\Phi\). For the case of our mirror the right-hand rule gives

\[
\oint \mathbf{E} \cdot d\mathbf{s} = E_z(z+dz)dx - E_z(z)dx = (dE_z/dz)dx + dz = -\frac{d\Phi}{dt} dx dz = -(d\Phi/dt)
\]

which leads immediately to Eq. (22). Similarly, the integral form of Ampère’s law \(\oint \mathbf{B} \cdot d\mathbf{s} = 4\pi I\) leads to Eq. (24).

12Anonymous, 1781.

Vacuum Tube Testing Outfit. The 1937 catalogue of the Central Scientific Company of Chicago shows this apparatus at $14.00 to “aid the elementary student in the usually confusing study of radio vacuum tube characteristics.” The student was able to obtain curves of the plate current as a function of the plate voltage for various values of the negative grid voltage. This one, in the Greenslade Collection, has a type 201A triode tube installed. The patent date is from 1935. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)