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Kinematics and dynamics of elastic rods

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A simple macroscopic theory of elastic rods is presented in which all assumptions but one are consistent with both Newtonian mechanics and special relativity. The one distinguishing assumption is the inertial equivalence of energy. While invariance of the theory under Lorentz transformations is proved, all physical consequences—including the stress and velocity dependence of the length and inertial mass of a rod as well as the velocities of sound through it—are derived and can be tested in any one inertial frame. Exact wave solutions of the basic equations are obtained for an idealized elastic material in which the velocity of sound is independent of amplitude. These solutions are used to account for the kinematics and dynamics of accelerated rods, including the time-dependent processes which result in their overall Lorentz contraction.

1. INTRODUCTION

In Newtonian mechanics, the length of an elastic rod changes with stress but is independent of its velocity, while its linear momentum changes with velocity but is independent of stress. When the inertial equivalence of energy is considered, length and momentum each depend on both stress and velocity. While the velocity dependence of these and other quantities can be determined by symmetry under Lorentz transformations, symmetry alone does not determine the time-dependent stresses and velocities of different parts of a body during acceleration. For these, a more specific theory of elastic bodies is needed.

Theories based on Newtonian mechanics have long been used to describe the kinematics and dynamics of elastic bodies at nonrelativistic speeds. Corresponding theories invariant under the Lorentz transformations of special relativity rather than the Galilean transformations of Newtonian physics, such as those presented by Synge, Møller, and others, have been used less frequently, probably because of their considerable complexity and the lack of experimental tests. Nevertheless, a study of the processes which take place in elastic bodies during acceleration, as described in any one inertial frame, can contribute to our understanding not only of elastic bodies but also of more general conservation laws and symmetry principles.

Here, we introduce a simple theory of moving elastic bodies and use it to account for the kinematics and dynamics of rods accelerated to any speed less than that of light. While symmetry of this theory under Lorentz transformations is proved, this need not be used in deriving the physical consequences of the theory in any one inertial frame. Of the five basic equations of this theory, four are consistent with both Newtonian mechanics and special relativity. The fifth, from which symmetry under Lorentz rather than Galilean transformations follows, is the inertial equivalence of energy; that is, to each quantity $E$ of energy, there corresponds an inertial mass $E/c^2$. All aspects of the theory can be directly compared with their Newtonian counterparts by substituting zero for $1/c^2$.

To focus on the most essential physical ideas, we consider only one-dimensional rods, and we assume that all deformations occur elastically and adiabatically, with no thermal or hysteresis effects. To facilitate the physical interpretation of the theory and comparisons with its Newtonian limit, we leave the factor $1/c^2$ explicit and separately specify the space and time components of space-time vectors and tensors.

Throughout this paper, we choose a single though arbitrary inertial frame for the description of all physical processes, just as we choose a single though arbitrary set of units. We will consistently leave this choice implicit, and refer to the velocity of a particle or the momentum and kinetic energy of a body without repeating each time that these are with respect to the arbitrarily chosen inertial frame.

To ensure that no hidden assumptions are used in deriving our basic results, and to separate their precise statement and proofs from more informal and intuitive discussions of their physical significance, we give five essentially self-contained theorems and mark the end of their proofs with a square, . We define $\theta$ and $\theta_i$ as differentiation with respect to space and time, and similarly $\theta_\nu$ and $\theta_i$ as differentiation with respect to stress $S$ and velocity $v$. While we consider only one-dimensional rods, we choose lowercase greek letters for quantities which in the three-dimensional case are scalars, lowercase latin letters for quantities which generalize to 3-vectors, and uppercase latin letters for those which generalize to second-rank tensors or $3 \times 3$ matrices.

2. BASIC QUANTITIES AND ASSUMPTIONS

In the theory to be presented, there are just two independent quantities defined at every point in an elastic body at each time. These are the stress $S$ and the velocity $v$. Figure 1 shows a rod moving with constant velocity $v$ while subject to a uniform compressional stress $S$ by a force $+S$ at its left end and $-S$ at its right end. In one dimension, the SI unit of stress is a newton, N, and in three dimensions it is a N/m². This theory will determine the time development of the stresses and velocities of all parts of an elastic body in terms of their values at any one time, the elastic properties of the material, and the values of any external forces.

In addition to the two independent quantities $S$ and $v$, four dependent ones are basic:

(i) $U$, the strain, whose increment $U_i = U_0$ at a point embedded in the material gives the factor $L_1$.

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equal to the density of the quantity times the velocity of the material, and a conductive part, equal to the flow past a point moving with the material,

\[ T = gv + S \quad \text{(2.4)} \]

and

\[ g = \rho v + Sv/c^2, \quad \text{(2.5)} \]

where \( 1/c^2 = 1.11265 \times 10^{-17} \) kg/J is the inertial equivalent of energy. The terms in Eqs. (2.4) and (2.5) have simple physical interpretations in the situation shown in Fig. 1. A rod under uniform stress \( S \) conducts linear momentum from one end to the other at the rate \( S \), giving the second term on the right of Eq. (2.4). A moving rod under stress also transmits energy from one end to the other at the rate \( Sv \), and the inertial equivalent \( Sv/c^2 \) of this energy flux gives the second term on the right of Eq. (2.5). This is the only place where the fundamental constant \( 1/c^2 \) enters our basic assumptions, Eqs. (2.1)–(2.5).

3. STRESS AND VELOCITY DEPENDENCE OF QUANTITIES

Equations (2.4) and (2.5) alone give the momentum density \( g \) and the momentum flux \( T \) as functions of stress \( S \), velocity \( v \), and inertia density \( \rho \). We now show that the stress and velocity dependence of \( \rho \) as well as of \( g \) and \( T \), and the strain \( U \) are uniquely determined by our general assumptions, Eqs. (2.1)–(2.5), together with a constitutive equation giving the mass density \( \rho = \rho \big|_{v=0} \) as a function of stress for the particular elastic material.

**Theorem 1.** For any differentiable function \( \mu \) of \( S \) with \( \mu + S/c^2 \geq 0 \), there is just one set of functions \( \rho \), \( g \), \( T \), and \( U \) of \( S \) and \( v \) which satisfy the boundary conditions \( \mu = \rho \big|_{v=0} \) and \( 0 = U \big|_{x=0} \) as well as Eqs. (2.1)–(2.5) for all \( \partial_x S \) and \( \partial_x v \). These are

\[ \rho = \frac{\mu + Sv^2/c^2}{1 - v^2/c^2}, \quad \text{(3.1a)} \]

\[ g = \frac{\mu v + Sv^2/c^2}{1 - v^2/c^2}, \quad \text{(3.1b)} \]

\[ T = \frac{\mu v^2 + S}{1 - v^2/c^2} \quad \text{(3.1c)} \]

and

\[ U = \frac{1}{2} \ln(1 - v^2/c^2) - \int_0^\mu \frac{d\mu}{\mu + S/c^2}. \quad \text{(3.2)} \]

**Proof.** Use the chain rule to express the \( x \) and \( t \) derivatives of \( \rho \), \( g \), \( T \), and \( U \) in terms of their \( S \) and \( v \) derivatives, for example, \( \partial_t U = \partial_x U \partial_t S + \partial_x U \partial_t v \). Substitute these into Eqs. (2.1)–(2.3) to get three linear equations in \( \partial_x S \), \( \partial_x v \), \( \partial_x v \), and \( \partial_x v \).
rod satisfies Hooke’s law $\partial_S^2L = 0$ if and only if

$$(\mu + S/c^2)\partial_S^2\mu = 2\partial_S\mu(\partial_S\mu + 1/c^2). \quad (3.6)$$

The general solution for this second-order differential equation is

$$\mu = \frac{\mu_0 + S^2/2\mu_0a^2c^2}{1 - S/\mu_0a^2c^2},$$

where $\mu_0 = \mu\big|_{v=0}$ is the mass density of an unstressed rod at rest and $a_0 = (\partial_S\mu_0)^{-1/2}|_{v=0}$ will be shown to be the speed of sound through an unstressed rod at rest. This result can also be obtained by adding the inertial equivalent $E/c^2$ of the elastic energy $E = L_0S^2/2\mu_0a^2$ to the unstressed mass $\mu_0L_0$ and dividing by the length $L = L_0(1 - S/\mu_0a^2c^2)$ of the rod under stress.

Theorem 1 shows that Eqs. (2.1)–(2.5) determine the velocity as well as the stress dependence of a rod. They give the $(1 - v^2/c^2)^{1/2}$ Lorentz contraction factor of an accelerating rod in any one inertial frame, rather than the ratio of lengths ascribed to the same rod by observers in different inertial frames. The stress and velocity dependence of the total inertial mass $\rho L$ and total momentum $gL$ is also uniquely determined by Eqs. (2.1)–(2.5) with appropriate boundary conditions, since $\rho$ and $g$ as well as $L$ are so determined.

4. SOUND VELOCIITES

In Newtonian physics, the inertial mass $\rho L$ of a rod is independent of both stress and velocity, its momentum $gL$ changes with velocity but not stress, and its length $L$ and strain $U$ change with stress but not velocity; however, since the speed of sound $a$ through matter at rest usually depends on stress, the velocities $w_+ = v \pm a$ of sound through moving matter depend on both stress $S$ and velocity $v$. When the inertial equivalence of energy is considered, all these quantities depend on both stress and velocity. Theorem 1 shows that Eqs. (2.1)–(2.5) together with the elastic properties of the material determine the stress and velocity dependence of $S$ and $V$, and now we show they determine the stress and velocity dependence of the velocities $w_{\pm}$ of sound as well.

We define a $2 \times 2$ matrix $W$ which gives the time derivatives of $S$ and $V$ as linear combinations of their space derivatives,

$$[\partial_S \partial_V] = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} \partial_S \mu \\ \partial_V \mu \end{bmatrix}. \quad (4.1)$$

The eigenvalues $w_{\pm}$ of the matrix $W$ are the velocities of sound, since when $S$ and $V$ depend only on $x - vt$, we have $\partial_T S = -w\partial_T \mu$ and $\partial_T V = -w\partial_T \mu$ for the eigenvectors of $W$ is defined as the acoustic impedance of the material.

Theorem 2. For any differentiable function $\mu$ of $S$ with $\mu + S/c^2 \neq 0$ and $\partial_S\mu > 1/c^2$, Eqs. (2.1)–(2.5) with the boundary condition $\mu = \mu\big|_{r=0}$
uniquely determine the stress and velocity dependence of the matrix $W$ in Eq. (4.1) for all $v^2 < c^2$, and it is

$$W = \frac{1}{\partial_\nu^2 \mu - v^2/c^4} \times \left[ \begin{array}{c} v(\partial_\nu \mu - 1/c^2) + \mu + S/c^2 \\ (1 - v^2/c^2)^2 \partial_\nu \mu + v(\partial_\nu \mu - 1/c^2) \end{array} \right]. \quad (4.2)$$

The eigenvalues of this $W$ are

$$w_k = \frac{v \pm a}{1 \pm av/c^2}, \quad (4.3)$$

where

$$a = (\partial_\nu \mu)^{-1/2}. \quad (4.4)$$

The ratio of the components of the eigenvectors of $W$ is

$$\frac{\partial_\nu S}{\partial_\nu \nu} = \frac{\partial_\nu S}{\partial_\nu T} = \pm \frac{\mu + S/c^2}{1 - v^2/c^2}. \quad (4.5)$$

Proof. Use the chain rule to express the $x$ and $t$ derivatives of $\rho$, $g$, and $T$ in Eqs. (2.2) and (2.3) in terms of their $S$ and $v$ derivatives, and express the result as

$$\left[ \begin{array}{cc} \partial_\nu \rho & \partial_\nu T \\ \partial_\nu g & \partial_\nu T \end{array} \right] \left[ \begin{array}{c} \partial_\nu S \\ \partial_\nu \nu \end{array} \right] = - \left[ \begin{array}{cc} \partial_\nu \rho & \partial_\nu T \\ \partial_\nu g & \partial_\nu T \end{array} \right] \frac{\partial_\nu S}{\partial_\nu \nu}. \quad (4.6)$$

This has a unique solution for $\partial_\nu S$ and $\partial_\nu \nu$ if and only if the determinant of their $2 \times 2$ coefficient matrix is not zero. Use Eqs. (3.1) to express the $S$ and $v$ derivatives of $\rho$, $g$, and $T$ in terms of $S$, $\nu$, $\mu$, and $\partial_\nu \mu$, so the determinant is $\partial_\nu \rho \partial_\nu g - \partial_\nu g \partial_\nu \rho = (\mu + S/c^2)(\partial_\nu \mu - v^2/c^2)(1 - v^2/c^2)^2$. Since the theorem assumes $\mu + S/c^2 \neq 0$, $\partial_\nu \mu > 1/c^2$, and $v^2 < c^2$, this determinant is nonzero and we can multiply on the left by the inverse matrix to obtain Eq. (4.1) with $W$ given by Eq. (4.2).

Since $\partial_\nu \mu$ is assumed to be positive, Eq. (4.4) defines a positive number $a$. Obtain the eigenvalues $w_\pm$ of $W$ as the roots of the characteristic equation for $W$, which in terms of $\alpha$ is

$$w^2(1 - \alpha^2/c^4) - 2w(1 - \alpha^2/c^2) + \alpha^2 = 0.$$ 

This factors as

$$[w(1 + \alpha/c^2) - v - \alpha][w(1 - \alpha/c^2) - v + \alpha] = 0,$$

from which Eq. (4.3) follows. The ratio $\partial_\nu S/\partial_\nu \nu$ is determined by the eigenvalue equation

$$\partial_\nu \alpha = \frac{1}{\alpha} \partial_\nu^2 \mu,$$

which with Eq. (3.5) gives

$$\frac{\partial_\nu T}{T} = \frac{\partial_\nu L}{L} = \frac{\partial_\nu \mu}{\alpha} = \frac{1}{\alpha} (\mu + S/c^2) + \frac{1}{\alpha} \partial_\nu^2 \mu.$$

Thus the period of a sound wave going back and forth through a rod is independent of stress if and only if

$$(\mu + S/c^2) + \partial_\nu^2 \mu = 2(\partial_\nu \mu)^2. \quad (4.6)$$

This agrees with Eq. (3.6) based on Hooke's law only in the Newtonian limit when the inertial equivalent of energy $1/c^2$ is neglected.

5. A GENERALIZATION OF HOOKE'S LAW

The stress dependence of mass density $\mu$ can be determined in several ways, for example, by measuring the length $L$ of a rod or the speed $a$ of sound through it as a function of stress and using Eq. (3.5) or (4.4). It can also be derived theoretically, at least in principle, by a quantum statistical model for the microscopic structure of the material. For most materials, only $\mu$ and $\partial_\nu \mu$ for small $S$ can be determined without exceeding elastic limits, but it is conceptually and computationally convenient to extrapolate from these to an idealized constitutive equation which gives $\mu$ as a function of $S$ for large stress as well. Two different such extrapolations are determined by Eqs. (3.6) and (4.6), the first based on the assumption that the change in length is proportional to stress, and the second on the stress independence of the period of a sound wave oscillating back and forth in a rod. We now consider a third extrapolation which is more useful than either of these for analyzing large-amplitude sound waves. It will be derived from the assumption that the velocity of a sound wave is independent of its amplitude. These three extrapolations are all equivalent to Hooke's law only in the Newtonian limit.

Consider two waves moving in the same direction, say
The velocity of each is given by Eq. (4.3), \( w_x = (a + v)/(1 + av/c^2) \), where \( v \) is the velocity of the material and \( a \) is the speed of sound, which depends on stress. Each wave produces fluctuations in both velocity and stress and these must have exactly opposite effects on \( w_x \) if neither wave is to change the velocity of the other. We now determine the stress dependence of mass density \( \mu \) that is necessary and sufficient for this cancellation to occur.

**Theorem 3.** Let \( \mu \) be a differentiable function of \( S \) and define \( a \) and \( w_x \) by Eqs. (4.3) and (4.4). Then \( \partial w_x / \partial S \) is 0 for all \( \partial \mu / \partial S \) and \( \partial \mu / \partial \nu \) satisfying Eq. (4.5) if and only if \( \mu \) satisfies

\[
(\mu + S/c^2)\partial_x^2 \mu = 2\partial_\nu \mu(\partial_x \mu - 1/c^2), \quad (5.1)
\]

The general solution of this equation is

\[
\mu = \frac{\mu_0 + S/c^2}{1 - S(a_0^2 - c^2)/\mu_0}, \quad (5.2)
\]

where \( \mu_0 = \mu \big|_{S=0} \) and \( a_0 = a \big|_{S=0} \). This function is differentiable and satisfies \( \mu + S/c^2 \neq 0 \) for all \( S \) in the interval

\[
-\frac{\mu_0 a_0 c}{1 + a_0/c^2} < S < -\frac{\mu_0 a_0 c^2}{1 - a_0/c^2}. \quad (5.3)
\]

For these \( S \),

\[
a = (\partial_\nu \mu)^{-1/2} = [1 - S(a_0^2 - c^2)/\mu_0]a_0 \quad (5.4)
\]

and \( 0 < a < c \).

**Proof.** Set to zero the \( x \) derivative of \( w_x \) given by Eq. (4.3) to get \( (1 - a_0^2/c^2)\partial_x^2 \pm (1 - v^2/c^2)\partial_\nu^2 = 0 \). Differentiate Eq. (4.4) with respect to \( x \) to get \( \partial_x a = -a(\partial_x^2 \mu)/\partial S/2 \), so that \( \partial_x a = 0 \) if and only if \( (1 - a_0^2/c^2)\partial_x^2 = \pm (1 - v^2/c^2)\partial_\nu^2 = 0 \). This holds for all \( \partial_x S \) and \( \partial_\nu \) satisfying Eq. (4.5) if and only if \( (1 - a_0^2/c^2) = a^2(\mu + S/c^2)\partial_x^2 \). Substitute \( \partial_x \mu = a^2 \) from Eq. (4.4) into this to get the differential equation (5.1). Differentiation verifies that the \( \mu \) of Eq. (5.2) is the general solution of this equation and gives the \( a \) of Eq. (5.4). As \( S \) approaches the positive limit \( \mu_0 a_0^3/(1 - a_0^2/c^2) \), \( \mu \) increases without bound and \( a \) goes to zero, while as \( S \) approaches the negative limit \( -\mu_0 a_0^3/(1 + a_0/c) \), \( \mu + S/c^2 \) goes to zero and \( a \) approaches \( c \).

From Eqs. (3.5) and (5.2) we obtain the stress dependence of the length of a rod for this idealized constitutive equation:

\[
L = L_0 \left( \frac{1 + 1/S/\mu_0 c^2}{1 + [2 - (a_0^2 - c^2)/\mu_0 S/\mu_0 c^2]^{1/2}} \right), \quad (5.6)
\]

Combining Eqs. (5.2) and (5.5) gives the stress dependence of the inertial mass,

\[
\frac{\mu L}{\mu_0 L_0} = \frac{1 + 1/S/\mu_0 c^2}{1 + [2 - (a_0^2 - c^2)/\mu_0 S/\mu_0 c^2]^{1/2}}, \quad (5.6)
\]

and combining Eqs. (5.4) and (5.5) gives the stress dependence of the period of acoustic oscillation,

\[
T = \frac{2L}{a} = \left( \frac{2L_0}{a_0} \right) \left( \frac{1}{\mu_0} \right) \left( \frac{1}{1 + [2 - (a_0^2 - c^2)/\mu_0 S/\mu_0 c^2]^{1/2}} \right). \quad (5.7)
\]

This period and the inertial mass given by Eq. (5.6) become independent of stress in the Newtonian limit when 0 replaces \( 1/c^2 \).

Figure 2 sketches the stress dependence of length given by Eq. (5.5), inertial mass given by Eq. (5.6), and the velocities of sound given by Eq. (5.4), together with the velocity dependence of these quantities and their Newtonian limits. The shapes of these curves depend only on the ratio \( a_0/c \), and the exceptionally large value of \( a_0/c = 0 \) was used in making these graphs to exhibit more clearly the differences between these stress dependencies and their Newtonian limits. As the stress approaches its positive limit \( \mu_0 a_0^3/(1 - a_0^2/c^2) \), the rod is compressed to arbitrarily small length, its inertial mass increases toward the limiting value \( \mu_0 L_0/(1 - a_0^2/c^2)^{1/2} \) which it has when moving with velocity \( a_0 \) while unstressed, the velocities of sound approach zero, and the period of acoustic oscillations approaches its minimum value \( (2L_0/a_0)(1 - a_0^2/c^2)^{1/2} \).

![Fig. 2. Dependence of length \( L \), inertial mass \( \mu_0 L_0 \), and the velocities of sound \( v \) on the stress \( S \) and velocity \( v \) of an elastic rod satisfying a generalization of Hooke's law. Dashed lines show the Newtonian limits.](image-url)
As the stress approaches its negative limit $-\mu_0a_0/c^2(1 + a_0/c^2)$, the length and inertial mass of a rod increase without bound, the velocities of sound approach $\pm c$, and the period of acoustic oscillations increases without bound.

The idealized constitutive equation (5.2) can also be expressed as $\mu S = (\mu - S/c^2 - \mu_0)a_0^2/(1 - a_0^2/c^2)$. Since it follows directly from Eqs. (3.1) that $\rho T = c^2 S$ and $\rho - T/c^2 = \mu - S/c^2$, we obtain an idealized constitutive equation for just $\rho$, $g$, and $T$,

$$\rho T - g^2 = \frac{(\rho - T/c^2 - \mu_0)a_0^2}{1 - a_0^2/c^2}$$ (5.7)

Equations (2.2), (2.3), and (5.7) give three equations for just $\rho$, $g$, and $T$, and mathematically the simplicity of this idealized constitutive equation follows from the linearity of Eq. (5.7) in the velocity-independent quantities $\rho T - g^2$ and $\rho - T/c^2$.

Exact wave solutions of Eqs. (2.1)–(2.5) and (5.2) are now readily obtained, since for this idealized constitutive equation the velocity of a sound wave is independent of its amplitude and waveform.

**Theorem 4.** Let $\mu_0$, $a_0$, and $v_0$ be numbers with $\mu_0 > 0$, $0 < a_0 < c$, and $v_0 < c^2$. Define $w = (a_0 + v_0)/(1 + a_0^2/c^2)$, let $f$ be any function of $x - wt$ bounded by $-1/a_0 < f < 1$, and define $\phi = f(1 + a_0^2/c^2) - (1 - f)/(1 - a_0^2/c^2)$ Then a solution to Eqs. (2.1)–(2.5) and (5.2) is

$$S = \frac{f a_0^2 \mu_0}{1 - f a_0^2/c^2},$$ (5.8a)

$$v = \frac{v_0 + fa_0}{1 + fa_0 v_0/c^2},$$ (5.8b)

$$\mu = \frac{\mu_0}{1 - f},$$ (5.8c)

$$U = \frac{1}{2} \ln \left(1 - \frac{v_0^2}{c^2}\right) + \ln \left(1 - f\right)/(1 + fa_0 v_0/c^2),$$ (5.8d)

$$\rho = \frac{1 + \phi}{1 - v_0^2/c^2 \mu_0},$$ (5.8e)

$$g = \frac{v_0 + \phi w}{1 - v_0^2/c^2 \mu_0},$$ (5.8f)

$$T = \frac{v_0^2 + \phi a_0^2}{1 - v_0^2/c^2 \mu_0}.$$ (5.8g)

From each of these solutions, another can be obtained by substituting $-a_0$ for $a_0$ throughout.

**Proof.** Since $\phi$ is a function of $f$, it too depends only on $x - wt$ so $\theta f_0 = -w \theta f_0$. Use this and Eqs. (5.8e)–(5.8g) to verify Eqs. (2.2) and (2.3). Verify Eqs. (2.4) and (2.5) by substituting for $S$, $v$, $\rho$, $g$, and $T$ from Eqs. (5.8) and using the definition of $\phi$ in terms of $f$. Finally, use $\theta f = -w \theta f$ and Eqs. (5.8b) and (5.8d) to verify Eq. (2.1).

The arbitrary function $f$ in Theorem 4 can be interpreted as an invariant and dimensionless amplitude for the wave. Equations (5.8a) and (5.8b) can be solved for $f$ in terms of $S$ or $v$ to give $f = S/(\mu_0 + S/c^2) a_0^2 = (v - v_0)/(1 - v_0^2/c^2) a_0^2$. When $f$ is small and the inertial equivalence of energy is neglected, then the fluctuations in strain produced by the wave equation $-f$, and $S = f \mu_0 a_0^2$, $v = \frac{v_0 + fa_0}{1 + fa_0 v_0/c^2}$, $\mu = \frac{\mu_0}{1 - f}$, $g = \frac{v_0 + \phi w}{1 - v_0^2/c^2 \mu_0}$, and $T = \frac{v_0^2 + \phi a_0^2}{1 - v_0^2/c^2 \mu_0}$.

### 6. ACCELERATING RODS

We now use Theorem 4 to describe quantitatively what happens when a constant force is applied to one end of an elastic rod which at $t_0 = 0$ is at rest and unstressed, as shown in the space–time diagram of Fig. 3. We assume an idealized material in which the velocity of sound is independent of amplitude.

Because the applied force is assumed constant, the total momentum of the rod increases linearly, but the distribution of this momentum is uneven and changes with time so that no part of the rod accelerates continuously as does its center of inertia. Instead, a sound wave starts at $t_0 = 0$ to travel down the rod, accelerating each part of the rod as it passes, leaving it with constant stress and velocity. At time $t_A$ when the wave front is reflected from the other end of the rod, the entire rod is under a stress equal to the applied force and it is all moving with the same velocity. As the wave front returns, it leaves behind an unstressed region moving at a higher velocity. In the adiabatic approximation, the wave continues to go back and forth, the stress at each point of the rod alternates between zero and the applied force, and the velocity at each point increases in a stepwise fashion. If the applied force is removed when the entire rod is unstressed, as at time $t_B$ or $t_D$ in Fig. 3, there will be no further changes in stress or velocity.

For a quantitative description of the first pass of the wave, we use Theorem 4 with $v_0 = 0$, $f_0 = 0$ for $x - wt > 0$, and $f = v_0^2/a_0$ for $x - wt < 0$, where $w = a_0$ is the velocity of the first wave. Substituting into Eqs. (5.8) gives the quantities at time $t_A$, after this wave has passed:

$$S_A = \frac{a_0^2 \mu_0}{1 - a_0^2/c^2},$$

$$v_A = v_0,$$

$$U_A = \ln(1 - v_0/a_0),$$

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The length $L_\Lambda$ of the rod at time $t_\Lambda$ is 

$$L_\Lambda = L_0 \exp(U_\Lambda - U_0) = L_0(1 - v/a_0).$$

The time required for the first wave to traverse the rod is $t_\Lambda = L_\Lambda/a_0$.

For the return wave, we match the stress and velocity at time $t_A$, using $v_\theta = 2v/(1 + v^2/c^2)$, $f = -v/a_0$ for $x - wt < L_0 - wt_A$, and $f = 0$ for $x - wt > L_0 - wt_A$, with

$$w = (v_\theta - a_0)(1 - a_\theta^2/c^2)$$

is the velocity of the return wave. Substituting into Eqs. (3.8) gives the quantities at time $t_B$, after the first acceleration cycle:

$$S_B = 0,$$

$$v_B = \frac{2v}{1 + v^2/c^2},$$

$$U_B = \ln\left(\frac{1 - v^2/c^2}{1 + v^2/c^2}\right),$$

$$\rho_B = \mu_0 \left(1 + \frac{v^2/c^2}{1 - v^2/c^2}\right)^2,$$

$$\sigma_B = 2\mu_0 \frac{v(1 + v^2/c^2)}{(1 - v^2/c^2)^2},$$

$$T_B = 4\mu_0 \frac{v^2}{(1 - v^2/c^2)^2}. $$

The length $L_B$ of the rod at time $t_B$ is 

$$L_B = L_\Lambda \exp(U_B - U_\Lambda) = L_0 \exp(U_B - U_0) = L_0(1 - v^2/c^2)/(1 + v^2/c^2).$$

The time $t_B - t_A$ is determined by $(t_B - t_A)(v - w_{AB}) = L_\Lambda = L_0(1 - v/a_0)$ and is $t_B - t_A = t_A(1 + v^2/c^2 - 2a_\theta^2/c^2)/(1 - v^2/c^2)$. In the Newtonian limit, $t_B - t_A = t_A$ and each traversal by the wave takes the same time. When the inertial equivalence of energy is considered, then when the rod is being pushed within its elastic limits, $0 < v < a_0$ and $t_B - t_A < t_A$ while, when it is pulled, $-c < v < 0$ and $t_B - t_A > t_A$.

Alternate kinematic and dynamic arguments can be used to check the results obtained with Theorem 4. For example, since the external force $F = S$ increases the total momentum of the rod at the rate $S$ and its total inertial mass at the rate $Fv/c^2$, conservation laws alone give $gL = St$ and $\mu L = \mu_0 L_0 + Svt/c^2$ at times $t_0 = 0$, $t_A$, and $t_B$.

Subsequent acceleration cycles can either be analyzed directly as was the first, or they can be obtained from the first by active Lorentz transformations to be considered in the next section. As long as the applied force remains constant, the magnitude of $f$ is the same in every cycle, $f = S/(\mu_0 + Sv/t/c^2)$. The velocity $v$ of the rod when all of it is stressed and the velocity $v_B$ at the end of each acceleration cycle are given in terms of $a_\theta$ and the velocity $v_\theta$ at the beginning of each cycle by

$$v = (f a_\theta + v_\theta)/(1 + f a_\theta v_\theta/c^2)$$

and $v_B = (f a_\theta + v_\theta)/(1 + f a_\theta v_\theta/c^2)$. The ratio of the unstressed lengths of the rod before and after each cycle is

$$L_1/L_2 = (v - v_\theta)/(v - v) = (1 - fa_\theta v_\theta/c^2)/(1 + fa_\theta v_\theta/c^2) = (1 - v^2/c^2)^{1/2}/(1 - v_\theta^2/c^2)^{1/2}.$$
For some quantities, such as stress, measurements made on state $A$ at the space–time point $P$ give the same result $S_A|_P$ as measurements made on new state $lA$ at the space–time point $lP$. For other quantities, such as the velocity $v$ of the material, $v_A|_P$ determines $v_{lA}|_l$ but these are not equal.

We now consider how a Lorentz transformation $l$ by a velocity shift $u$ changes each of the basic quantities of this theory. The coordinates of the points $P$ and $lP$, in the same coordinate system, are related by

$$x_{lP} = \frac{x_P + lP}{1 - u^2/c^2}$$

and

$$t_{lP} = \frac{t_P + x_P u/c^2}{1 - u^2/c^2}.$$  \hspace{1cm} (7.1a, 7.1b)

If $-u$ is substituted for $u$, these equations give the inverse transformation $l^{-1}$, and if 0 is substituted for the inertial equivalent of energy $1/c^2$, these give a Galilean transformation.

**Theorem 5.** Let $S_A$, $v_A$, $\rho_A$, $T_A$, and $U_A$ be one solution to Eqs. (2.1)–(2.5) and let $u$ be any number with $u^2 < c^2$. Then a new solution $S_{lA}$, $v_{lA}$, $\rho_{lA}$, $g_{lA}$, $T_{lA}$, and $U_{lA}$ of these same equations is given by

$$S_{lA} = S_A, \quad v_{lA} = \frac{v_A + u}{1 + uv_A/c^2}.$$  \hspace{1cm} (7.2a)

$$\rho_{lA} = \frac{\rho_A + 2g_Au + T_Au^2/c^2}{1 - u^2/c^2},$$  \hspace{1cm} (7.2b)

$$g_{lA} = \frac{\rho_Au + g_A(1 + u^2/c^2) + T_Au/c^2}{1 - u^2/c^2},$$  \hspace{1cm} (7.2c)

$$T_{lA} = \frac{\rho_Au^2 + 2g_Au + T_A}{1 - u^2/c^2},$$  \hspace{1cm} (7.2d)

$$U_{lA} = U_A + \ln \frac{1 - u^2/c^2}{1 + uv_A/c^2},$$  \hspace{1cm} (7.2e)

where the coordinates of points $P$ and $lP$ are related by Eqs. (7.1).

**Proof.** Use Eqs. (7.2) alone to derive

$$(gV + S - T)_A = \left[ gV + S - T + \left( \rho v + Sv/c^2 - g \right) u \right]_A |_P$$

and

$$(\rho u + Sv/c^2 - g)_{lA} = \left[ \rho u + Sv/c^2 - g \right]_A |_P.$$  \hspace{1cm} (7.2f)

From these it follows that, if $S$, $v$, $\rho$, $g$, and $T$ for state $A$ satisfy Eqs. (2.4) and (2.5), they do for state $lA$ as well.

When considering the differential equations (2.1)–(2.3), it is convenient to extend the definition of the Lorentz transformation $l$ so that it permutes functions over space–time as well as space–time points. For any function $f$ of $x$ and $t$, define a new function $f_l$ by $f_l|_{lP} = f|_P$ for all $P$: that is, the value of the new function $f_l$ at the new point $lP$ always equals the value of the old function $f$ at the old point $P$. It follows immediately that $l$ preserves sums and products of functions; that is, $l(f + g) = l(f) + l(g)$ and $l(fg) = (l(f))(l(g))$ for any two functions $f$ and $g$ of $x$ and $t$. The transformation $l$ does not commute with differentiation with respect to $x$ and $t$, since Eqs. (7.1) and the chain rule for differentiation imply the operator identities

$$\partial_t l = l(\partial_t - u \partial_x),$$  \hspace{1cm} (7.3a)

and

$$\partial_x l = l(\partial_x - u \partial_t / c^2).$$  \hspace{1cm} (7.3b)

Use these and Eqs. (7.2) alone to derive

$$\partial_t \rho_{lA} + \partial_x g_{lA} = \frac{l[\partial_t \rho_A + \partial_x g_A + (\partial_t T_A + \partial_x S_A)u/c^2]}{(1 - u^2/c^2)^{1/2}},$$

and

$$\partial_t g_{lA} + \partial_x T_{lA} = \frac{l[\partial_t g_A + \partial_x T_A + (\partial_t \rho_A + \partial_x g_A)u]}{(1 - u^2/c^2)^{1/2}}.$$  \hspace{1cm} (7.3f)

Hence, when $\rho$, $g$, and $T$ for state $A$ satisfy Eqs. (2.2) and (2.3), they do so for state $lA$ as well. Finally, use Eqs. (7.2) and (7.3) alone to derive

$$(\partial_t + v_{lA} \partial_x) U_{lA} = - \partial_x v_{lA} + l \left( \partial_t + v_A \partial_x \right) U_A - \frac{\partial x v_A}{1 + uv_A/c^2} \left( 1 - u^2/c^2 \right)^{1/2},$$

so that, when $v_A$ and $U_A$ satisfy Eqs. (2.1), $v_{lA}$ and $U_{lA}$ do so as well.

For a simple and useful example, we make a Lorentz transformation by $v$ on a stressed rod at rest, so that $S_A = S$, $v_A = 0$, $\rho_A = \mu$, $g_A = 0$, $T_A = S$ and $U_A = -S d\mu (\mu + S/c^2)$, and the final state $lA$ is described by Eqs. (3.1) and (3.2).

The transformation of most of the basic quantities in this theory are familiar ones; $S$ and $\mu$ are scalars, $\rho$, $g$, and $T$ are the components of a symmetric rank two space–time tensor, and $v$ is the ratio between the components of a space–time vector. However, the strain $U$ and the length $L$ of a rod are not as simply related to space–
time vectors and tensors. While quantities such as \( \rho, g, \) and \( T \) which transform linearly can always be decomposed into scalars, vectors, and tensors of other rank, this is not the case for quantities such as strain \( U \) which undergo nonlinear transformations.

The proof of Theorem 5 shows that transforming any solution of Eq. (2.1) alone gives a new solution, whether or not the old one satisfies any of the other basic equations. Hence, the set of all solutions or the solution set for Eq. (2.1) is invariant under Lorentz transformations even though this equation is not written in covariant form. While none of the other four basic equations (2.2)–(2.5) has a solution set that is invariant under Lorentz transformations, the intersection of those for Eqs. (2.2) and (2.3) is invariant, as is the intersection of solution sets for Eqs. (2.4) and (2.5).

8. SUMMARY AND CONCLUSIONS

The macroscopic theory of elastic rods that has been presented assumes five basic equations, (2.1)–(2.5), relating stress \( S \), velocity of the material \( v \), strain \( U \), inertial density \( \rho \), inertial flux and momentum density \( g \), and momentum flux \( T \). Section 2 presents these equations and the intuitive physical significance of each. Theorem 1 shows that the stress and velocity dependence of \( \rho, g, T, \) and \( U \) is uniquely determined by these assumptions together with appropriate boundary conditions. From these, the usual relativistic results for the velocity dependence of the length and inertial mass of a rod are derived.

Theorem 2 shows that, in this theory, the values and space derivatives of \( S \) and \( v \) at any point uniquely determine their time derivatives, and from this the velocities \( (v \pm c)/2 \) of sound through moving matter are derived, where \( c \) is the speed of sound through matter at rest. Theorem 3 determines the stress dependence of mass density that is necessary and sufficient for the velocity of sound waves to be independent of amplitude. In Newtonian mechanics, this reduces to Hooke’s law. Theorem 4 gives exact, finite-amplitude wave solutions for these equations, with the values for \( S, v, \rho, g, T, \) and \( U \) at all parts of the wave. Theorem 5 shows that from any solution of the basic equations, regardless of the elastic properties of the material, new ones can be obtained by active Lorentz transformations which change the physical situation as described in any one inertial frame. This establishes the Lorentz invariance of the theory even though the basic equations are not written in manifestly covariant form.

This approach to the study of elastic bodies has certain advantages even in situations when relativistic effects are small. It provides a detailed picture for the localized densities and flows of conserved quantities in elastic objects subject to stress and acceleration, and this often contributes to an understanding of physical processes at any speed. It describes different physical processes in a single, arbitrarily chosen inertial frame by using just one coordinate system and set of units rather than transforming among equivalent descriptions of the same physical processes.

Many of our results are certainly well known from special relativity, including the velocity dependence of the length, inertia, and period of oscillation of an accelerating rod, and the relativistic velocity addition formula. All that is new for these here is that they are derived in any one inertial frame from assumptions in which the fundamental constant \( 1/c^2 \) enters only in the one term of Eq. (2.5), where it associates a momentum density \( S/vc^2 \) to the energy flux \( S \).

In another paper,\(^5\) certain other occurrences of the speed of light \( c \) in physics, including those in Maxwell’s equations of electromagnetism, were also derived from assumptions in which the only \( 1/c^2 \) terms were identified with the inertial equivalence of energy. These results support the conjecture that with adequate specific theories all occurrences of the speed of light \( c \) in physics can be derived from the inertial equivalent of energy \( 1/c^2 \).

In addition to providing an alternate derivation of well-known results and a more detailed model for following relativistic processes in any one inertial frame, this theory gives certain new results. For example, it shows that certain conditions which each determine the stress dependence of mass density, while all equivalent to Hooke’s law in Newtonian mechanics, give different generalizations of it when the inertial equivalence of energy is taken into account. Three of these are (i) deformation proportional to stress, \( \partial_x^2 \mu = 0 \) or [Eq. (3.6)]

\[(\mu + S/c^2)\partial_x^2 \mu = 2\partial_x \mu(\partial_x \mu + 1/2c^2),\]

(ii) stress independence of the period of acoustic oscillations, \( \partial_x(L/\alpha) = 0 \) or [Eq. (4.7)],

\[(\mu + S/c^2)\partial_x^2 \mu = 2(\partial_x \mu)^2,\]

and (iii) amplitude independence of the velocity of sound waves [Eq. (5.1)],

\[(\mu + S/c^2)\partial_x^2 \mu = 2\partial_x \mu(\partial_x \mu - 1/c^2).\]

This theory also gives more information about the dynamics of elastic deformation and sound wave propagation than is determined by Lorentz invariance alone.

While only one-dimensional rods have been considered in this paper, the basic equations (2.1)–(2.5) generalize to three space dimensions. In this case, additional constitutive equations are needed to relate shear stresses to the deformations they produce in each elastic material, since the stress dependence of mass density no longer uniquely determines the elastic properties of material as it does for the one-dimensional case.

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\(^5\)W. C. Davidson, Found. Phys. (to be published).