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A new model of the solar cycle

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ABSTRACT
A new model of the solar magnetic activity cycle is proposed. The model is based on the symmetries of a rotating star, and describes the interaction between dipole and quadrupole fields in the weakly non-linear regime. The model describes not only pure dipole and quadrupole dynamos, but also the transitions with increasing dynamo number to mixed polarity states, and in appropriate regimes to aperiodic states which are highly suggestive of prolonged activity minima such as the Maunder minimum.

Key words: Sun: activity – Sun: magnetic fields – Sun: rotation – sunspots.

1 INTRODUCTION

The study of the solar magnetic cycle dates from the time of Galileo. It is now known that the cycle has an approximate 22-yr period but variable amplitude. In particular, during certain prolonged periods, now known as the Maunder or grand minima, the number of sunspots has decreased dramatically, and sunspots may have been absent altogether (Weiss 1994). A number of more or less realistic models have been proposed, both for the basic solar cycle and for its amplitude 'vacillation'. These models are of two types: the first relies on numerical integration of the full MHD equations, usually with a parametrized α-effect and field loss via magnetic buoyancy. In recent years this approach has been pursued particularly by A. Brandenburg and colleagues (see, e.g., Brandenburg et al. 1989a,b). The second approach, pursued by N. O. Weiss and colleagues (Weiss, Cattaneo & Jones 1984; Jennings 1991; Jennings & Weiss 1991; Weiss 1993, 1994; Tobias, Weiss & Kirk 1995), has focused on model systems that attempt to capture the essential physics of dynamo action while preserving the correct spatial symmetries of the problem, but otherwise makes no attempt at quantitative predictions (see also Platt, Spiegel & Tresser 1993). In fact, these two approaches are somewhat complementary and, given the parametrization of the full equations, neither approach is at present capable of detailed predictions (cf. Hoyng 1990; Weiss 1994).

The present paper is in the spirit of the second approach. It takes as its starting point the observation that the solar dynamo does not, in fact, operate in a pure multipole state (Tang, Howard & Adkins 1984; Stenflo & Vogel 1986; Visozo & Ballester 1990). Such states will be referred to as mixed parity or non-symmetric states. The slow modulation of the solar cycle can then be thought of as a low-frequency oscillation between the dipole and quadrupole components. Such an oscillation could be periodic (if major activity minima occur periodically), or aperiodic as in the Sun. The paper focuses on the interaction of the dipole and quadrupole modes, and studies the most general equations describing such a mode interaction that are consistent with the symmetries of a rotating star. The modes involved are either axisymmetric or non-axisymmetric; the latter drift in the azimuthal direction as well as in latitude. Such modes are entirely natural in systems of this type (cf. Net, Mercader & Knobloch 1995). The equations are then truncated at third order. Such a truncation requires that the two modes bifurcate at nearby dynamo numbers. If this is so, the truncated equations provide a quantitatively exact description of the mode interaction. If the two modes bifurcate at dynamo numbers that differ substantially, the truncated equations should still provide a qualitative description of the interaction.

In addition to the observational support for mixed parity dynamos, many dynamo models also show such behaviour. Moreover, such models show that the first two modes to become unstable are typically the dipole and quadrupole modes (e.g. Brandenburg et al. 1989a, b; Jennings & Weiss 1991), providing additional support for studying their non-linear interaction. The equations describing this interaction contain a relatively small number of free coefficients, and these can be varied to explore the possible dynamics resulting from the interaction. These coefficients can be computed by standard techniques from whatever basic partial differential equations are of interest. In the absence of such calculations, the model does not have quantitative predictive power; it can, however, be used to shed light both on the solar cycle and on the dynamics observed in the numerical simulations, and can do so more simply than using partial differential equations. It is this point of view that is adopted here. The model is based on the ideas put forward by Knobloch

(1994a) and suggests a plausible mechanism for the occurrence of Maunder minima.

2 THE MODEL

In the following we let $z_o(t)$ and $z_e(t)$ be the (complex) amplitudes of the toroidal magnetic field in the dipole and quadrupole states. In the former the toroidal field is antisymmetric with respect to the equator, while in the latter it is symmetric. These fields are thus odd and even with respect to reflection in the equator. In spherical polar coordinates (r, θ, ϕ) the toroidal field may then be written in the form

$$B_t(r, \theta, \phi) = \mathrm{Re}\{z_o e^{im\phi} f_o(r, \theta) + z_e e^{im\phi} f_e(r, \theta)\} + \text{higher order terms}, \quad (1)$$

where $f_{o,e}$ denote the spatial eigenfunctions of the two modes, and m is the azimuthal wavenumber. The two eigenfunctions have the following symmetry properties: $f_o(r, \pi-\theta) = -f_o(r, \theta)$, $f_e(r, \pi-\theta) = f_e(r, \theta)$. There is a similar expression for the poloidal field with amplitudes that are related to (z_o, z_e) by the solution of the linear dynamo problem. Consequently, the dynamics of both fields is fully determined once the time evolution of (z_o, z_e) is known. Note that we have assumed that both competing modes have the same non-zero azimuthal wavenumber m. This is a plausible assumption, since the dynamo number is expected to be a strong function of m. It follows from rotation and reflection invariance of the problem (i.e., invariance under $\phi \to \phi + \phi_0$ and $\theta \to \pi - \theta$) that the amplitude equations must be equivariant with respect to

$$\text{rotation: } (z_o, z_e) \to e^{im\phi_0}(z_o, z_e), \quad (2a)$$

$$\text{reflection: } (z_o, z_e) \to (-z_o, z_e). \quad (2b)$$

The equations for (z_o, z_e) thus take the form, truncated at third order,

$$\frac{dz_o}{dt} = (\mu + i\omega) z_o + K_1 |z_o|^2 z_o + K_2 |z_e|^2 z_o + K_3 \bar{z}_o z_e^2, \quad (3a)$$

$$\frac{dz_e}{dt} = (\mu' + i\omega') z_e + K_1' |z_e|^2 z_e + K_2' |z_o|^2 z_e + K_3' \bar{z}_e z_o^2. \quad (3b)$$

Here $\mu + i\omega$ and $\mu' + i\omega'$ are the growth rates and frequencies of the two modes, and K_j, K_j' are complex coefficients. The assumption that $m \neq 0$ is not essential to the derivation of these equations. Similar equations follow even when $m = 0$ (axisymmetric modes), provided that the frequencies of the two modes are sufficiently near one another that the problem is captured by an unfolding of the 1:1 temporal resonance (Landsberg & Knobloch 1994). Such situations do, in fact, arise in axisymmetric dynamo models (Jennings & Weiss 1991). It should be noted, however, that the $m = 0$ case is special in that the initial instability need not be a Hopf bifurcation; in such systems dynamo waves need not be present. In the general case with two incommensurate frequencies ω, ω', equations (3a) and (3b) can be simplified even further by so-called normal form transformations. Since the validity, in parameter space, of such normal form equations is more limited than that of the original system, we choose to work with equations (3a) and (3b). In describing the solutions of these equations and their relation to the dynamics of the magnetic field we shall retain the terminology dipole and quadrupole when referring to the amplitudes z_o and z_e, even though it does not apply, strictly speaking, to non-axisymmetric fields; the terminology odd/even is more precise.

In the following we set, without loss of generality, $\mu = \lambda + \Delta\lambda$, $\mu' = \lambda - \Delta\lambda$, and treat λ as the bifurcation parameter. Thus the dipole state sets in when $\lambda = -\Delta\lambda$, and precedes the onset of the quadrupole state at $\lambda = \Delta\lambda$ provided that $\Delta\lambda > 0$. In general, these pure parity states are the only ones that bifurcate from the non-magnetic state described by $z_o = z_e = 0$, although, as discussed below, in appropriate circumstances an additional *quasi-periodic* solution can also bifurcate from this state. Note that the pure parity (non-axisymmetric) modes take the form of dynamo waves that travel not only in the azimuthal direction but also in latitude. For example, if $z_e = 0$ and we write $f_o = |f_o| e^{i\Phi}$, then equations (1) and (3a) show that $B_t = |z_o||f_o(r, \theta)| \times \cos[\Omega t + m\phi + \Phi(r, \theta)]$. Here Ω is the frequency of the *non-linear* dipole dynamo, $|\Omega - \omega| = \mathcal{O}(\lambda + \Delta\lambda)$. For fixed r, ϕ, this expression describes the usual 'butterfly' diagram for such a dynamo. In addition, it shows that the pure parity modes are necessarily *rotating* waves, so that spatial rotations are equivalent to time evolution. It is also easy to check the general result that only the first mode to become unstable can be stable and then only when it bifurcates supercritically (Iooss & Joseph 1980).

3 DYNAMICS OF THE MODEL

Equations (3a) and (3b) depend on six complex coefficients, in addition to the parameters specifying the linear problem. To illustrate the dynamical behaviour exhibited by equations (3a) and (3b) and reduce the number of parameters, we set $K_1 = K_1'$, $K_2 = K_2'$, $K_3 = K_3'$. This is, in fact, a consistent assumption whenever the two modes are nearly degenerate, i.e., whenever $f_o \sim f_e$, $0 < \theta < \pi/2$. In the following we suppose that the first mode to become unstable is the dipole mode, with the quadrupole mode following at a slightly larger λ (dynamo number). Although it is not necessary, we also suppose that $\mathrm{Re}\, K_1 < 0$, i.e., that both modes bifurcate supercritically. Equations (3a) and (3b) have a number of possible solutions, which we now describe and relate to the magnetic field. Time-independent solutions of the form $(|z_o|, |z_e|) = (|z_o|, 0)$ and $(0, |z_e|)$ correspond, respectively, to pure dipole and quadrupole modes. These are therefore periodic dynamos in which dynamo waves propagate towards the equator in a periodic fashion with well-defined symmetries relative to the equator. When $|\omega - \omega'| = \mathcal{O}(\Delta\lambda)$, the non-linear terms promote frequency locking between the *non-linear* frequencies Ω, Ω' of the two competing modes, resulting in periodic dynamos with a constant mixture of both dipole and quadrupole modes. These states correspond to time-independent solutions of equations (3a) and (3b) of the form $(|z_o|, |z_e|)$, $|z_o z_e| \neq 0$, and appear in pitch-fork bifurcations from the pure parity states. In addition, there are two types of oscillatory solutions $(|z_o(t)|, |z_e(t)|)$, ones that oscillate about a pure parity mode, and ones that oscillate about a mixed parity mode. These correspond to dynamos that are quasi-periodic, since the new oscillation frequency is superimposed on the basic dynamo frequency ω. As we discuss below, such solutions come about through *secondary* bifurcations, and hence their frequency is always small relative to the dynamo frequency. This is because the new frequency is related to the *amplitude* of the solutions at which the bifurcation takes place. This frequency has therefore nothing to do with the linear beat frequency between the two modes, nor does the growth rate of the linear modes have any bearing on the nature of the non-linear state that will ensue. As also discussed below, the symmetric oscillations are usually produced through secondary Hopf bifurcations from a pure parity state, or through so-called gluing or global bifurcations. Finally, there are circumstances under which there is a bifurcation to such quasi-periodic oscillations directly from the trivial state (Swift 1988). In this case the new frequency starts from zero, rather than from a small but finite value, and increases with increasing λ. All of these solutions, except the last, can be identified with the various solutions found by Brandenburg et al. (1989a, b) and Schmitt & Schüssler (1989) in their numerical integration of the mean field dynamo equations with α-quenching and flux loss through magnetic buoyancy. The corresponding 'butterfly' diagrams are readily constructed, since within the present theory the spatial structure of the modes is given simply by the dipole and quadrupole eigenfunctions. Thus all that is necessary is to take the solutions $(z_o(t),\ z_e(t))$ and multiply them by the appropriate eigenfunctions, plotting the results for fixed r, ϕ in the $(\theta,\ t)$-plane using the representation (1), cf. Dangelmayr, Knobloch & Wegelin (1991). However, in contrast to the problem studied by Dangelmayr et al., the dynamo problem admits waves travelling in one direction only, towards the equator. Consequently, we take

$$f_{o,e}(\theta)=e^{(\gamma-ik)\theta}\sin 2\theta, \qquad 0<\theta<\frac{\pi}{2} \tag{4a}$$

$$f_{o,e}(\theta)=\pm e^{(\gamma-ik)(\pi-\theta)}\sin 2\theta, \qquad \frac{\pi}{2}<\theta<\pi, \tag{4b}$$

and use these trial eigenfunctions to construct butterfly diagrams illustrative of the different possible solutions of equations (3a) and (3b). These eigenfunctions are the simplest ones that capture the directionality (equatorward if $k>0$, poleward if $k<0$) and approximate shape of the unstable dynamo modes. Here γ is a real parameter related to the group velocity of the waves, and the r, ϕ dependence of the eigenfunctions has been omitted. The eigenfunctions are continuous at $\theta=\pi/2$ where they vanish, and fall off towards either pole provided that $\gamma>0$. In Fig. 1 we show examples of butterfly diagrams for pure dipole and quadrupole fields constructed using these eigenfunctions. These are, respectively, antisymmetric and symmetric in the equator. We have chosen to represent these (and subsequent) solutions in terms of a sequence of time slices, instead of the more usual contour plots, in order to be able to represent states with more complex time-dependence as well. These solutions represent fully non-linear solutions of the dynamo problem, since they solve equations (3a) and (3b) for the indicated values of the coefficients K_1, K_2 and K_3. To generate these solutions we have chosen the mode-splitting parameter $\Delta\lambda$ arbitrarily (here $\Delta\lambda=0.0001$), assuming that the frequency difference $\omega-\omega'$ depends linearly on $\Delta\lambda$. It should be clear that the dynamics of equation (3a) and (3b) depend only on this difference. The actual frequencies ω, ω' are required only for the construction of the corresponding butterfly

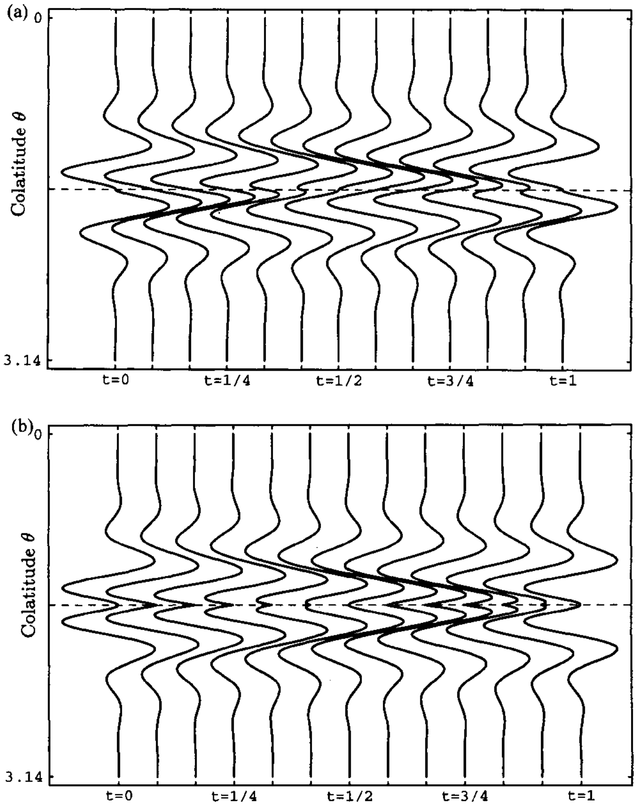

Figure 1. Stable dynamo solutions in the form of (a) a dipole, and (b) a quadrupole. The states are represented by a sequence of equally spaced time slices $B_t(\theta,\ t_n)$, $n=1,\ \ldots,$ extending over one period of the dynamo cycle. Here θ is the colatitude, and the dashed line denotes the equator. The parameter values for the dipole solution shown are $(\mu,\ \mu',\ \omega,\ \omega',\ K_1,\ K_2,\ K_3)=(0.0004,\ 0.0002,\ 6.283,\ 6.263,\ -1.0+1.0i,\ -2.0+1.0i,\ -0.3+2.0i)$, i.e., $\lambda=0.0003$, $\Delta\lambda=0.0001$, $\omega-\omega'=0.02$. The quadrupole solution was obtained by interchanging the primed and unprimed parameters. The eigenfunction parameters are $k=11.0$, $\gamma=5.0$.

diagrams. In order to construct these, one needs to choose, in addition to the absolute frequencies, the eigenfunction parameters k and γ. For the case shown in Fig. 1 we have used $k=11.0$, $\gamma=5.0$, so that most of the activity is confined to a $\pm 30°$ strip around the equator, as observed in the Sun. The figures show clearly that the dynamo waves propagate towards the equator, either out of phase or in phase in the two hemispheres. Because no effort has been made to choose trial eigenfunctions that are differentiable at the equator, the quadrupole solution exhibits some unphysical behaviour at $\theta=\pi/2$. If one used exact eigenfunctions, as in the model problem worked out by Worledge et al. (1995), this artefact would be absent, although at the expense of a significantly more complex expression in place of equations (4a) and (4b). Even in the absence of a theoretical prediction, the appropriate eigenfunctions can in principle be determined from observational data and used to elucidate the appearance of the toroidal field corresponding to other solutions of equations (3a) and (3b). In view of the fact that the butterfly diagrams inevitably depend on the choice of the eigenfunctions, we represent in Fig. 2 the corresponding solutions in terms of the time series $z_o(t)$ and $z_e(t)$. Note that, in spite of the distinct and apparently complex spatial dependence, both solutions have *identical* sinusoidal dynamics. In fact, if viewed in terms of the projection on to the amplitudes $|z_o|$ and $|z_e|$, these solutions are just *fixed* points. Both of these representations provide an eigenfunction-independent characterization of these simple solutions. In Fig. 3 we show the slightly more complex mixed parity solution, represented by a fixed point of the form $(|z_o|, |z_e|)$, $|z_o z_e| \neq 0$. Fig. 4 shows the corresponding time series $(z_o(t), z_e(t))$. Observe that the two (complex) amplitudes have the same (non-linear) frequency, and a fixed phase relation. All these solutions satisfy equations (3a) and (3b), and their stability can be determined by linearizing these equations about these solutions.

In Fig. 5 we show stable *quasi-periodic* solutions. These solutions are, respectively, symmetric and asymmetric with respect to the equator. In the representation of Fig. 5 the solutions have been strobed with the basic dynamo frequency, leaving only the modulation frequency. The figures present equally spaced time slices within the modulation period. The equatorward motion of the dynamo waves cannot be seen in these figures, which focus on the long-time behaviour of the solutions. In order to characterize these solutions in an eigenfunction-independent way, we introduce, following Swift (1988), the (real) coordinates (u, v, w) defined by

$$u + iv = 2z_o\bar{z}_e, \tag{5a}$$

$$w = |z_e|^2 - |z_o|^2. \tag{5b}$$

The use of these coordinates eliminates the (non-linear) dynamo frequency from the dynamics while retaining information about the amplitudes and relative phase of the dipole

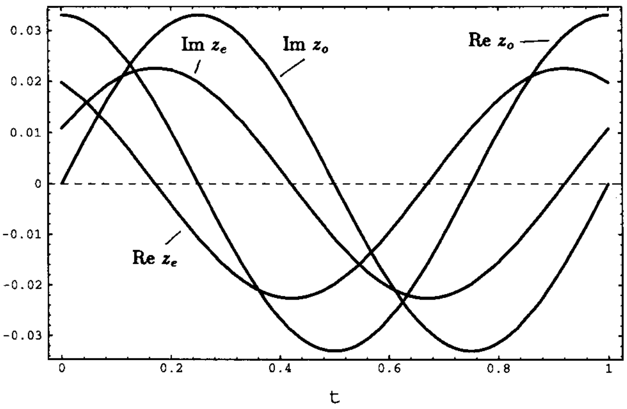

Figure 4. The time series $z_o(t)$ and $z_e(t)$ used to construct Fig. 3.

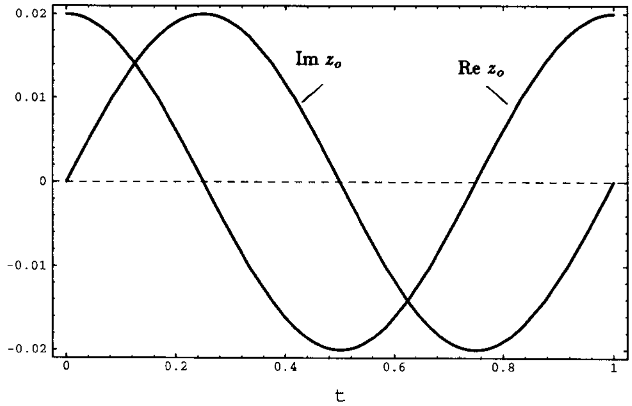

Figure 2. The time series Re $z_o(t)$ and Im $z_o(t)$ used to construct Fig. 1(a). Fig. 1(b) is constructed from identical time series for z_e.

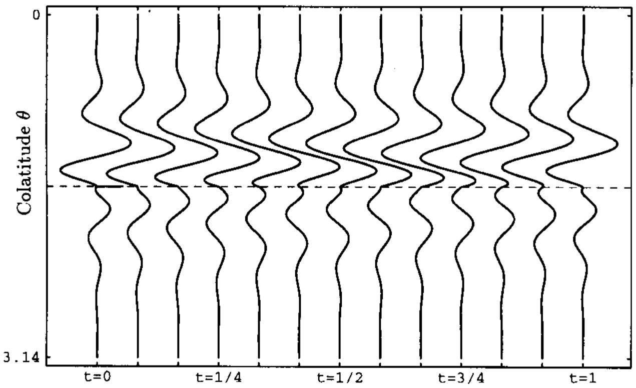

Figure 3. A mixed parity state for $(\mu, \mu', \omega, \omega', K_1, K_2, K_3) = (0.04, 0.02, 6.27, 6.29, -36.05+20.30i, -0.9187-16.62i, -1.0)$, i.e., $\lambda = 0.03$, $\Delta\lambda = 0.01$, $\omega - \omega' = -0.02$. The eigenfunction parameters are $k = 11.0$, $\gamma = 4.0$.

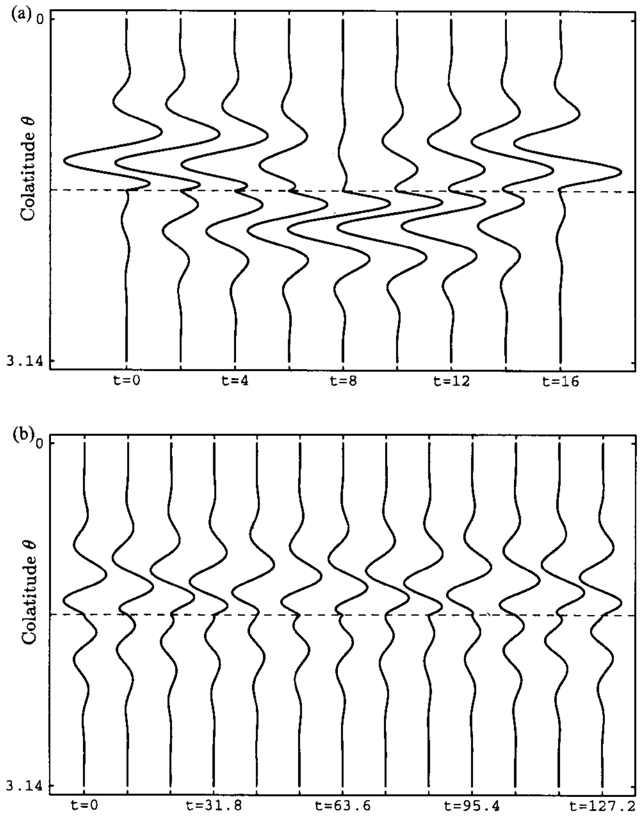

Figure 5. Modulated (two-frequency) dynamo states. (a) A stable symmetric state for $(\mu, \mu', \omega, \omega', K_1, K_2, K_3) = (0.16, 0.20, 6.5108, 6.0556, -1.0+1.0i, -1.0-0.3i, 1.0+0.5i)$, i.e., $\lambda = 0.18$, $\Delta\lambda = -0.02$, $\omega - \omega' = 0.4552$; (b) a stable non-symmetric state for $(0.11, 0.09, 6.2838, 6.2826, -0.875+1.0i, -1.015+1.0i, 0.02+0.5i)$, i.e., $\lambda = 0.1$, $\Delta\lambda = 0.01$, $\omega - \omega' = 0.0012$. The eigenfunction parameters are $k = 11.0$, $\gamma = 4.0$.

and quadrupole modes. Consequently, the quasi-periodic solutions shown in Fig. 5 correspond to simple limit cycles in the (u, v, w) variables. We show such limit cycles projected on the (v, w) plane in Fig. 6. To interpret this figure, note that a pure dipole state corresponds to the fixed point $(0, 0, w)$ with $w<0$, while the fixed point $(0, 0, w)$ with $w>0$ describes a pure quadrupole. Note also that the reflection symmetry in the equator, $(z_o, z_e) \rightarrow (-z_o, z_e)$, translates into the symmetry $(u, v, w) \rightarrow (-u, -v, w)$ in the new variables. Consequently, a symmetric quasi-periodic dynamo (Fig. 5a) corresponds to a limit cycle in the (v, w) plane which is symmetric under $v \rightarrow -v$ (Fig. 6a), while an asymmetric quasi-periodic dynamo corresponds to one of a *pair* of limit cycles related by reflection in $v=0$ (Fig. 6b). In contrast, the projection of the dynamics onto the variables $(|z_o|, |z_e|)$ completely hides their symmetry properties by ignoring the relative phase of the modes (see Figs 6c and d).

Analysis of equations (3a) and (3b) shows that as λ is varied for fixed $\Delta\lambda$ the pair of asymmetric cycles can form a double homoclinic connection to the fixed point corresponding to the dipole state; this connection then breaks, forming a symmetric limit cycle (Landsberg & Knobloch 1994). In Fig. 7 we show the result of solving equations (3a) and (3b) for λ just prior to this *gluing* bifurcation. The solutions are represented both in the (v, w) plane as in Fig. 6, but also in terms of the variables

$$E \equiv \sqrt{u^2+v^2+w^2} = |z_o|^2 + |z_e|^2, \tag{6a}$$

$$P \equiv w/E, \qquad -1 \leq P \leq 1, \tag{6b}$$

introduced by Brandenburg et al. (1989a,b) to analyse the results of their direct simulations of the partial differential equations for an α–Ω dynamo with a quenched α-effect. The variable E provides a measure of the magnetic field energy in the dynamo, while the quantity P measures the preference of one polarity over the other. Specifically when $P=-1$ the field is dipolar, while when $P=1$ it is quadrupolar. In both cases the proximity to the homoclinic connection results in a highly non-linear time series $E(t)$ and $P(t)$, spending long periods of time in a nearly pure dipole state, before making a brief and abrupt excursion in which the quadrupole mode is briefly excited. These brief periods are correlated with a significant drop in the magnetic energy E. This solution is evidently closely related to that found by Brandenburg et al. (1989b, fig. 5), who tentatively identify it with the Maunder-like grand minima in the solar cycle. In the following we refer to such solutions as type I grand minima. Note, however, that neither $E(t)$ nor $P(t)$ reveal the symmetry of the resulting dynamo state. Moreover, as shown in Figs 6(a) and (c), a stable symmetric quasi-periodic state can coexist with a stable mixed parity (periodic) state, showing that the dynamo can operate under identical conditions in one of several different modes. In these cases the 'initial conditions' determine the actual state realized.

Equations (3a) and (3b) also exhibit more complicated and perhaps more interesting behaviour, in which both amplitudes $|z_o|$, $|z_e|$ spend a significant time near zero before making a large-amplitude excursion. These solutions also describe strongly non-linear modulation of the dynamo

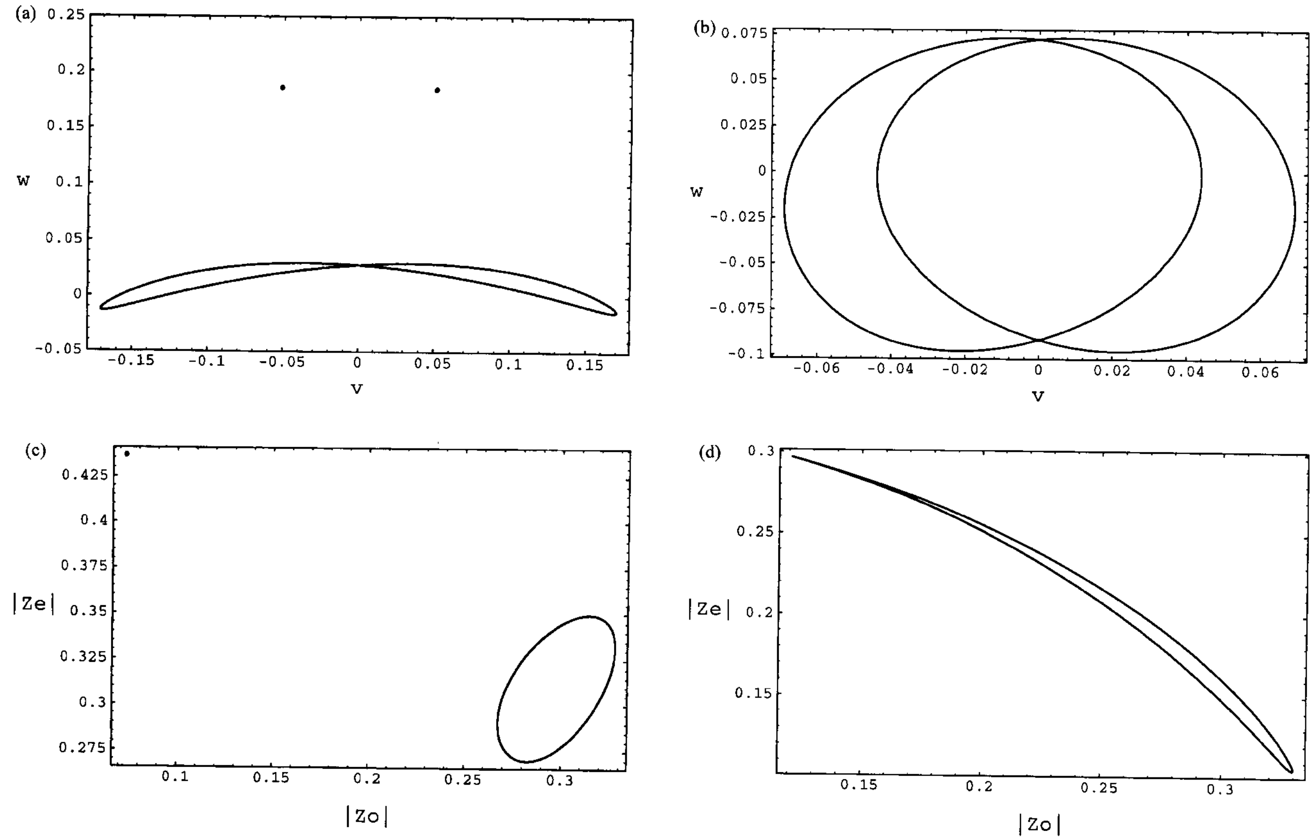

Figure 6. The limit cycles in the (v, w) and $(|z_o|, |z_e|)$ planes corresponding to Figs 5(a) and (b). Note that in (a) the symmetric modulated state coexists with a stable mixed parity periodic dynamo, corresponding to one of a pair of fixed points (cf. Fig. c).

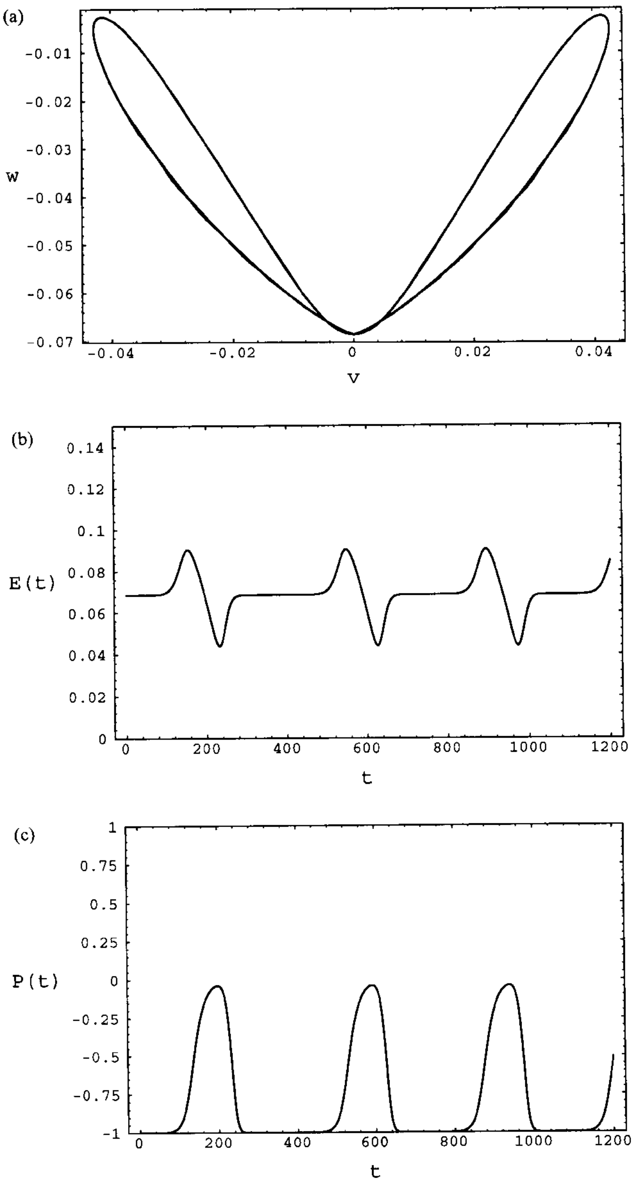

Figure 7. (a) The limit cycles in the (v, w) plane just prior to the gluing bifurcation, describing dynamos with type I grand minima; (b) and (c) The corresponding time series $E(t)$ and $P(t)$. The parameter values are $(\mu, \mu', \omega, \omega', K_1, K_2, K_3) = (0.0685942, 0.0357942, 6.24319, 6.32319, -1.0+1.0i, -1.0-0.3i, 1.0+0.5i)$, i.e., $\lambda = 0.0521942$, $\Delta\lambda = 0.0164$, $\omega - \omega' = -0.08$. The corresponding figures just after the gluing bifurcation are essentially indistinguishable, except that in (a) the trajectory now traverses both lobes of the limit cycle.

cycle, but this time it is the energy E that remains small over long times. Such states represent a second and distinct mechanism for producing Maunder-like grand minima. Figs 8 and 9 show that these type II minima can occur either periodically or irregularly, though always with a well-defined characteristic period. In this respect the present model differs from that put forward by Platt et al. (1993). In fact, the observational data indicate both that there is a well-defined interval between successive minima (208 yr) and that their form tends to be similar (Weiss 1994). The solutions shown in Figs 8 and 9 are highly non-linear, and are characterized

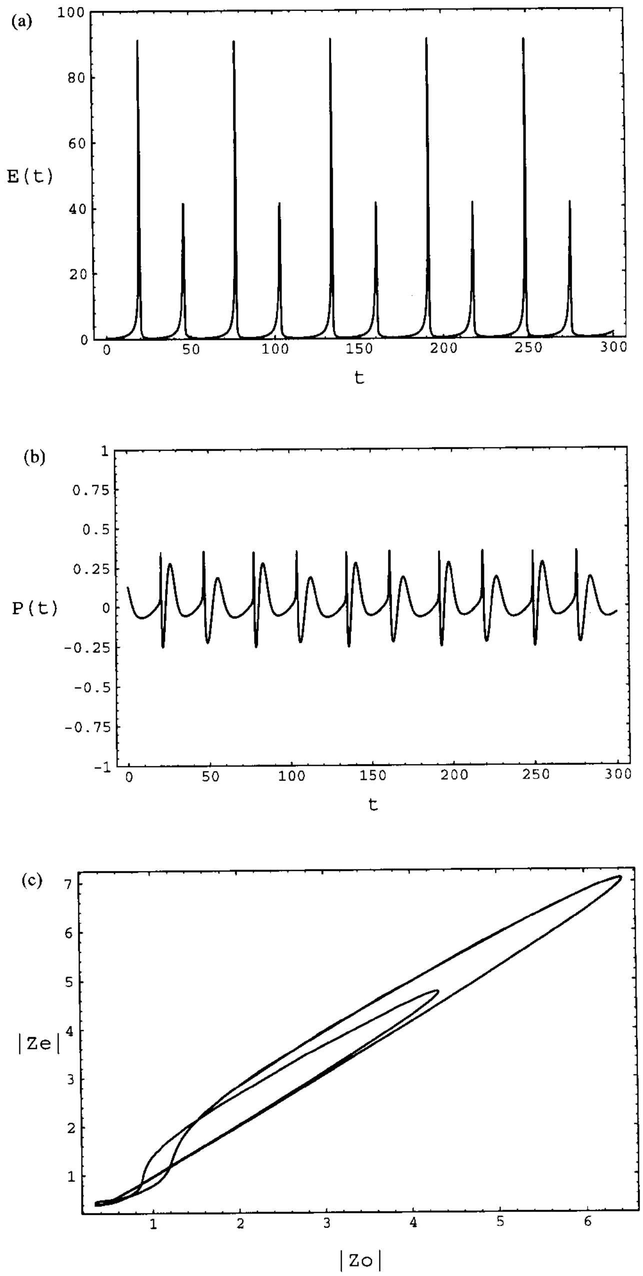

Figure 8. Solutions exhibiting periodic, type II grand minima. (a) The time series $E(t)$; (b) the time series $P(t)$; and (c) the projection on to the reduced phase space $(|z_o|, |z_e|)$. The parameters are $(\mu, \mu', \omega, \omega', K_1, K_2, K_3) = (0.035, 0.165, 6.323, 6.243, -0.95+1.5i, 0.5-1.0i, 0.5+0.5i)$, i.e., $\lambda = 0.1$, $\Delta\lambda = -0.065$, $\omega - \omega' = 0.08$.

by an exponential build up in the energy E of the magnetic field, followed by an abrupt collapse. At the same time, the dominant polarity of the field undergoes an abrupt reversal. Note that, unlike the solutions described in Fig. 7, the characteristic asymmetry of this new class of solutions resides in the time series $E(t)$ as opposed to $P(t)$. Consequently, these new solutions describe Maunder-like minima in which the field oscillates between fields containing both dipole and quadrupole components, resulting in a dynamo cycle that is superficially more complicated. Unlike the type I

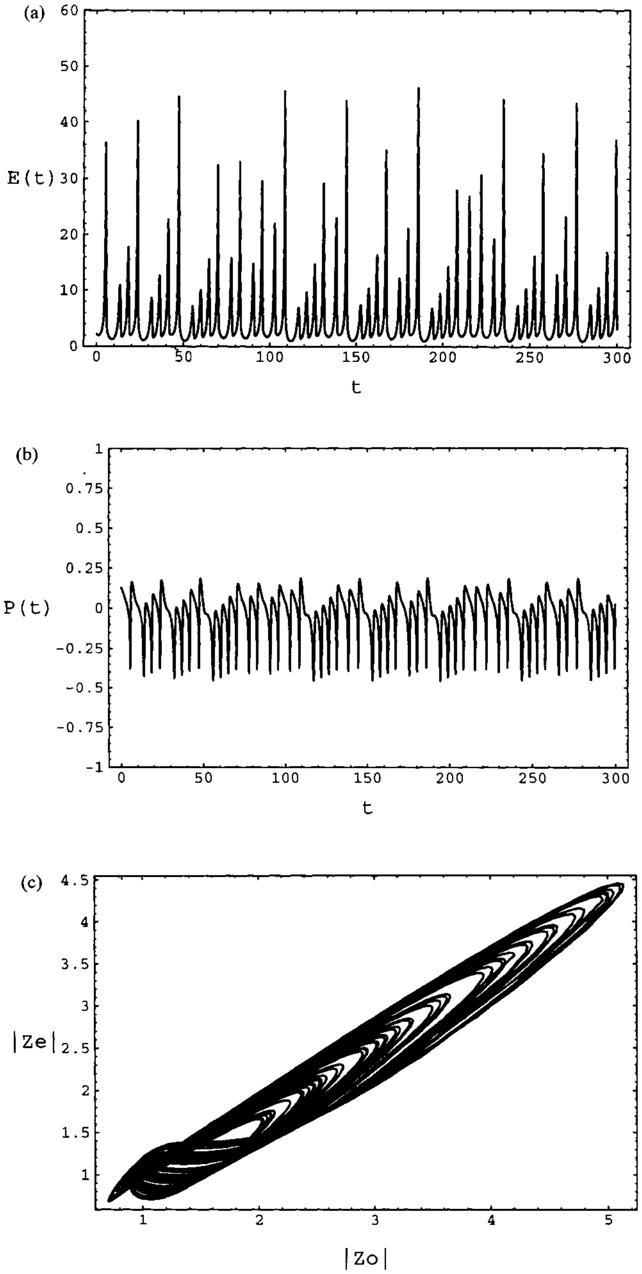

Figure 9. Solutions exhibiting (apparently) aperiodic, type II grand minima. (a) The time series $E(t)$; (b) the time series $P(t)$; and (c) the projection on to the reduced phase space $(|z_o|, |z_e|)$. The parameters are $(\mu, \mu', \omega, \omega', K_1, K_2, K_3)=(0.48, -0.08, 6.123, 6.443, -0.9+3.0i, 0.5+0.5i, 0.5+0.5i)$, i.e., $\lambda=0.2$, $\Delta\lambda=0.28$, $\omega-\omega'=-0.32$.

solutions, the type II states are apparently not directly associated with the proximity to a global bifurcation.

Yet other types of solutions can be obtained by increasing $|\omega-\omega'|$. For example, for large enough $|\omega-\omega'|$ the steady state $(|z_o|, |z_e|)$ will describe a quasi-periodic oscillation as the two (non-linear) frequencies Ω, Ω' unlock. Such oscillations resemble 'beating' familiar from linear problems, with the beat period corresponding to the period of the resulting amplitude vacillation, and can be easily distinguished from the two-frequency oscillations present when $|\omega-\omega'|$ is small. This is because in the former case the amplitudes (z_o, z_e) oscillate about a zero mean, while in the latter case they oscillate about a non-zero amplitude (and about a fixed phase as well).

It is interesting and instructive to compare the solutions described above with those obtained by Brandenburg et al. (1989a,b; see also Schmitt & Schüssler 1989) by direct simulation of the partial differential equations for an α–Ω dynamo with a quenched α-effect. The calculations by these authors (cf. Brandenburg et al. 1989b, fig. 5) reveal a scenario of the type described in Figs 6 and 7, i.e., the presence (for appropriate values of the dynamo number) of highly non-linear oscillations about a mixed parity state, in which the system spends most of the time in a state of nearly pure parity (dipole) containing the most energy, while making brief periodic excursions into the competing (quadrupole) state. The oscillations exhibit the same type of asymmetry in $P(t)$, with a slower rise and a rather more rapid fall-off, as shown in Fig. 7(c). Although Brandenburg et al. do not discuss the spatial symmetry of their solution, we believe on the basis of our investigation of equations (3a) and (3b) that the appearance of solutions of this type heralds a change in the symmetry of the cycle from a non-symmetric to a symmetric (quasi-periodic) dynamo. For other values of the dynamo number the observed solutions look more like that shown in our Fig. 6(b), indicating that the quasi-periodic dynamo is, in fact, asymmetric (cf. Brandenburg et al. 1989b, fig. 6). Note that the invariant circles in the Poincaré sections used by Brandenburg et al. to describe their quasi-periodic solutions are directly comparable to the limit cycles in the (v, w)-plane shown in Figs 6 and 7. The use by these authors of the full solution instead of its envelope results in inessential complexity and one that makes the understanding of the dynamics more difficult. For example, fig. 6 of Brandenburg et al. (1989b) is of the type shown in our Fig. 6(b), except that in our Fig. 6(b) the basic frequency of the dynamo waves has been removed. A further study of the case described in Figs 6 and 7 of this paper indicates that the symmetric quasi-periodic state created in the gluing bifurcation disappears in a *frequency locking* bifurcation that creates a pair of stable and a pair of unstable limit cycles on an invariant torus (Landsberg & Knobloch 1994, fig. 7). Each of the new limit cycles describes a mixed parity periodic dynamo, and all coexist with a pair of unrelated mixed parity states (cf. Fig. 6c) as well as the two pure parity states. Such a picture appears to be consistent with that summarized in fig. 12 of Brandenburg et al. (1989b). Brandenburg et al. find no evidence for the type II grand minima, or for the presence of chaotic dynamos. This is undoubtedly because of their inability to explore the parameter space; it is well known that the gluing bifurcation described in Figs 6 and 7 can, under appropriate conditions on the eigenvalues of the pure mode, lead to a homoclinic explosion, much as occurs in the well-known Lorenz equations (cf. Rucklidge & Matthews 1995). Equations (3a) and (3b) facilitate greatly the exploration of parameter space, and can be used to locate regions of non-periodic modulation.

4 CONCLUSION

The model proposed here is based on the rotation and reflection invariance of a rotating star. The former is responsible for the fact that all primary dynamo modes that break azimuthal symmetry precess, i.e., for the fact that all such modes are produced in Hopf bifurcations leading to rotating waves (cf. Ecke, Zhang & Knobloch 1992; Knobloch 1994a). The latter can be shown to imply that each mode is either odd or even with respect to the equator. The resulting amplitude equations describing the interaction of the first two such modes to become unstable as the dynamo number is increased are related to the equations put forward by Knobloch (1994a) to describe the solar cycle in terms of the amplitudes of dynamo waves travelling towards the equator in the northern and southern hemispheres. In particular, if we denote these amplitudes by v, $\overline{w}$, we see that $z_o = v - \overline{w}$, $z_e = v + \overline{w}$. The reflection symmetry $\theta \to \pi - \theta$ takes (v, w) to $(\overline{w}, \overline{v})$. In these variables equations (3a) and (3b) become

$$\frac{dv}{dt} = (\lambda + i\omega)v + \Delta\overline{w} + a|w|^2 v + b(|v|^2 + |w|^2)v + c\overline{v}\overline{w}^2, \quad (7a)$$

$$\frac{dw}{dt} = (\lambda - i\omega)w + \overline{\Delta}\overline{v} + \overline{a}|v|^2 w + \overline{b}(|v|^2 + |w|^2)w + \overline{c}\overline{v}^2\overline{w}, \quad (7b)$$

where $a = K_1 - K_2 - 3K_3$, $b = K_1 + K_2 + K_3$ and $c = K_1 - K_2 + K_3$, and we have again ignored the difference between K_j and K_j', $j = 1, 2, 3$. These equations differ from those suggested by Knobloch (1994a) only in the presence of the new terms $(\overline{v}\overline{w}^2, \overline{v}^2\overline{w})$. Consequently, it comes as no surprise that the dynamics of equations (3a) and (3b) bear substantial resemblance to that of equations (7a) and (7b) with $c = 0$: in addition to the two primary modes, equations (7a) and (7b) with $c = 0$ exhibit secondary bifurcations to single frequency mixed parity states and to quasi-periodic states in which the amplitudes $|v|$, $|w|$ oscillate with a slow frequency either about a pure mode ($|v| = |w|$) or about a mixed mode ($|v| \neq |w|$). Such oscillations persist only for an interval of dynamo numbers, and with increasing λ typically give way to the mixed parity states, either via a secondary Hopf bifurcation or a global bifurcation at which the new oscillation frequency vanishes. These transitions are typically hysteretic. See Dangelmayr & Knobloch (1991) and Knobloch (1994a) for more details. Indeed, in appropriate circumstances, the resulting oscillations will be chaotic (Knobloch 1994b; Hirschberg & Knobloch 1995), leading to a non-periodic dynamo. It is therefore not surprising that similar behaviour is also found in equations (3a) and (3b). However, this model also exhibits behaviour very much like the Maunder minima in the solar cycle, resulting either from a global bifurcation involving a pure dipole mode, or in the case of type II minima from the proximity to the origin (i.e., the unmagnetized state). The type II behaviour is apparently absent from equations (7a) and (7b) when $c = 0$. Both types of solutions are naturally identified with cycles in which the activity fluctuates over long time-scales (cf. Brandenburg et al. 1989b).

The model equations introduced here enable us to discuss with ease not only the possible dynamical states of any dynamo model sharing the symmetries of a rotating star but also the transitions between them, simply by varying the parameters and coefficients in equations (3a) and (3b). Such a study, though at present incomplete, allows us to obtain an understanding of the *possible* behaviour of such systems. The resulting model describes the possible interactions between dipole and quadrupole modes, both when the dipole sets in first ($\Delta\lambda > 0$) and when the quadrupole sets in first ($\Delta\lambda < 0$). We have seen that secondary instabilities of these pure parity states typically involve mixed parity states. These can be either single-frequency or two-frequency states, depending on the nature of the bifurcation creating them. We have also seen that there are substantial regions in the coefficient space of equations (3a) and (3b) in which the variation of a single parameter, the bifurcation parameter λ, will result in strongly non-linear amplitude modulation, and have identified at least three distinct mechanisms leading to such modulation. All of this behaviour is of codimension-one, i.e., it is accessible by varying a single parameter, the dynamo number λ. Such a detailed study would not be possible in models based on partial differential equations. In fact, the basic message of this paper is that even the simplest non-linear dynamo models consistent with the symmetries of a rotating star have such a bewildering variety of solutions that even a concerted effort has failed thus far to provide a complete description (cf. Dangelmayr & Knobloch 1991; Landsberg & Knobloch 1994; Hirschberg & Knobloch 1995).

In relying entirely on the symmetries of the problem the approach adopted in this paper strips the dynamo problem to its absolute essentials. In this respect it differs from earlier studies of this type which relied on models based on a drastic truncation of some dynamo equations (see, e.g., Weiss 1993). Such truncations have been found very helpful in interpreting dynamo behaviour. On the other hand, they are also limited in that they are not completely general. In the present approach we replace uncertainties in modelling various physical processes (and even uncertainty in which processes should be included!) by the freedom to select a relatively small number of coefficients in the amplitude equations. This approach, called astromathematics by Spiegel (1994), offers the best opportunity for exploring the possible dynamics that are available to dynamo models, and provides a viable explanation of Maunder minima. However, even these undoubtedly simplest possible equations appear to describe such a range of possible behaviour that a complete understanding of possible dynamo behaviour may remain elusive (cf. Hirschberg & Knobloch 1995).

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